Relativistic Chasles' theorem and the conjugacy classes of the inhomogeneous Lorentz group

E. Minguzzi*

Abstract

This work is devoted to the relativistic generalization of Chasles' theorem, namely to the proof that every proper orthochronous isometry of Minkowski spacetime, which sends some point to its chronological future, is generated through the frame displacement of an observer which moves with constant acceleration and constant angular velocity. The acceleration and angular velocity can be chosen either aligned or perpendicular, and in the latter case the angular velocity can be chosen equal or smaller than the acceleration. We start reviewing the classical Euler's and Chasles' theorems both in the Lie algebra and group versions. We recall the relativistic generalization of Euler's theorem and observe that every (infinitesimal) transformation can be recovered from information of algebraic and geometric type, the former being identified with the conjugacy class and the latter with some additional geometric ingredients (the screw axis in the usual non-relativistic version). Then the proper orthochronous inhomogeneous Lorentz Lie group is studied in detail. We prove its exponentiality and identify a causal semigroup and the corresponding Lie cone. Through the identification of new Ad-invariants we classify the conjugacy classes, and show that those which admit a causal representative have special physical significance. These results imply a classification of the inequivalent Killing vector fields of Minkowski spacetime which we express through simple representatives. Finally, we arrive at the mentioned generalization of Chasles' theorem.

Contents

*Dipartimento di Matematica Applicata "G. Sansone", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy. E-mail: ettore.minguzzi@unifi.it

1 Introduction

A rigid movement is an orientation preserving isometry of Euclidean space. A classical theorem by Euler states that every rigid movement admitting a fixed point can be accomplished through a rotation around some axis passing through the point. This result was generalized by Mozzi and Chasles, [7] who proved that in the general case in which no fixed point is required, the rigid movement can be accomplished through a rotation around some axis combined with a translation parallel to the axis. The composition of these two movements can be accomplished with a single screw or helical motion.

Mathematically, Euler's and Chasles' theorems establish the existence of a certain type of representative for each conjugacy class of the group $SO(3)$ and $ISO(3)$, respectively. The conjugacy transformation represents a change of frame, thus at the geometrical level the choice of a convenient representative corresponds to the choice of a convenient frame.

A related problem is that of finding the orbits of the adjoint (Ad) action of $SO(3)$ on its Lie algebra $\mathfrak{so}(3)$. The motivation is essentially the same: we wish to select a simple element of the orbit on $\mathfrak{so}(3)$ so as to read with ease the physical content of the infinitesimal transformation represented by the Lie algebra element. Usually the infinitesimal versions of Euler's and Chasles' theorem are regarded as special cases of their finite counterparts. The finite version can also be deduced from the infinitesimal one. The proof in this direction is essentially equivalent to the proof that the Lie group $SO(3)$ is exponential.

Ultimately, each Lie algebra element is a vector field and, in Chasles' case, it can be represented with a characteristic screw flow around a special line called *instantaneous axis of rotation*. The infinitesimal formulation of Chasles' theorem became the starting point of *Screw Theory*, a formulation of rigid body mechanics which unifies in the concept of screw the rotational and translational degrees of freedom of rigid bodies [4, 10, 28, 37, 25].

Euler's theorem was generalized to Minkowski space M by several authors [45, 1, 23, 35, 36, 39]. This problem is essentially equivalent to that of classifying the conjugacy classes of the proper orthochronous Lorentz group.

In this work we generalize Chasles' theorem by selecting a convenient representative for each conjugacy class of the inhomogeneous proper orthochronous Lorentz group. We identify the type of geometric data which is required in order to recover the original transformation. The simple form of the representatives will simplify the interpretation and, in particular, will allow us to prove a result which we can conveniently formulate as (we shall give precise definitions of all the terms involved, see Theorems 4.20 and 4.21)

Theorem 1.1. *Every proper orthochronous isometry of Minkowski spacetime, which sends some point to its chronological future, can be accomplished through the frame dragging of spacetime points, where the frame is that of an observer which moves with constant angular velocity and constant acceleration for some proper time interval. The observer can be chosen so that the acceleration and angular velocity are either aligned or perpendicular. In the latter case the angular velocity can be chosen no greater than the acceleration.*

Finally, there are two cases. If the observer's motion is of pure rotation, the proper time interval of motion duration and the angular velocity are uniquely determined, while if the observer's motion cannot be chosen to be a pure rotation, then the proper time interval can be chosen arbitrarily, and after this choice the modules of the acceleration and angular velocity are uniquely determined (Eqs. (28)-(31)).

Up to the freedom in the time duration, the acceleration and angular velocity are uniquely determined, thus they can be regarded as genuine characteristics of the isometry. With respect to the classical Chasles' theorem, here we need to impose a casuality condition, indeed, space translations are not generated by observer's motions while they satisfy the other hypothesis. As a consequence, we shall need some results on the way causality reflects itself on the Lie algebra. This will be done identifying a causal Lie semigroup and studying the corresponding Lie cone.

This paper is organized as follows.

In section 2 we recall the classical Euler's and Chasles' theorems, both in the Lie group and Lie algebra versions. We notice here that in order to recover the original transformation we need information of algebraic and geometric type, the former being identified with the conjugacy class (Ad-orbit, in the Lie algebra case) and the latter being identified with the screw axis. We also introduce the screw product on the Lie algebra as its generalization will provide a new Ad-invariant for the relativistic case.

In section 3 we study the Lorentz group introducing the usual Ad-invariants for the Lie algebra, and recalling the classification of the Lie orbits and conjugacy classes. We also identify the geometric data needed to recover the full (infinitesimal) transformation.

In section 4 we come to the inhomogeneous Lorentz group. In section 4.1 we introduce a causal semigroup of $ISO(1,3)^{\uparrow}$, showing its connection with isometries which send some point to its causal future. In section 4.2 we introduce our conventions and clarify the physical meaning of the Lie algebra generators. This section will be essential for the correct interpretation of subsequent results.

In particular we explain the importance of linear combinations of the form $\vec{a} \cdot \vec{K} + \vec{\omega} \cdot \vec{J} + H$, where the translational generators $\vec{P}$ do not appear. Indeed, we interpret these combinations as the allowed generators for the observer's motion. The relativistic Chasles' theorem will ask not only to prove that the generic transformation is the exponential of some infinitesimal generator, a fact proved in section 4.3, but also that the generator is of the mentioned form up to conjugacy.

In section 4.4 we introduce a set of Ad-invariants which allow us to completely classify the orbits of the adjoint action of $ISO(1,3)^{\uparrow}$, on $\mathfrak{iso}(1,3)$. This classification implies a classification of the inequivalent Killing fields of Minkowski spacetime. We clarify the relation between our classification and a slightly coarser one previously obtained by T. Barbot [5].

In section 4.5 we introduce the Lie cone of the causal semigroup. We answer the following question: given two frames (bases) in spacetime, with the application point of the latter in the chronological future of the former, is it always possible to regard them as the initial and final states of a comoving base attached to an observer which rotates and accelerates with constant angular velocity and acceleration for some proper time interval? The answer is negative unless the last frame is contained in a spacetime cone which is narrower than the light cone and which depends on the required angular velocity and acceleration.

In section 4.6 we show that the Lie Ad-orbits can be given a causal character depending on whether some representative belongs to the causal Lie cone. The causal orbits are in a way reminiscent of the classification of elementary particles (indeed, at least for finite groups, there is a bijection between conjugacy classes and irreducible linear representations). With theorem 4.20 we obtain the relativistic generalization of Chasles' theorem, in the Lie algebra formulation. Finally, in section 4.7 we give the group version.

Concerning our conventions, the indices i, j, k, take the values $1, 2, 3$, while the indices a, b, c or α, β, γ, take the values $0, 1, 2, 3$. We adopt the Einstein summation convention, and our signature for the Minkowski metric η_{ab} is $(-,+,+,+)$. A vector v is causal (timelike) if $\eta(v, v) \le 0$ (resp. < 0) and $v \neq 0$. The vector is nonspacelike if it is causal or $v = 0$. A vector is lightlike (null) if it is causal (resp. nonspacelike) but not timelike. The chronological future $I^+(x)$ of $x \in M$ is made by all the points that can be reached from x following future directed (f.d.) timelike curves. The causal future is denoted $J^+(x)$ and includes x plus all the point that can be reached from x following f.d. causal curves. For shortness, we shall sometimes use the word *direction* when referring to a 1-dimensional subspace of a vector space. We use units such that $c = 1$, where c is the speed of light. The subset symbol $\subset$ is reflexive, i.e. $X \subset X$.

For background on the inhomogeneous Lorentz group the reader might consult [6, 44, 38].

2 Euler's and Chasles' theorems

Let us formulate Chasles' theorem in mathematical language. Let E be the Euclidean space. This means that E is an affine space modeled over a 3-dimensional vector space $(V, \cdot, or)$, endowed with a positive definite scalar product $\cdot : V \times V \to \mathbb{R}$, and orientation or. A reference frame is a choice of origin $o \in E$ plus a positive oriented orthonormal base $\{\vec{e}_i, i = 1, 2, 3\}$ of V. Given a reference frame, every point $p \in E$ can be written in a unique way in terms of coordinates as follows $p = o + x^i \vec{e}_i$. The coordinate vector belonging to $\mathbb{R}^3$ will be denoted using a bar, e.g. $\bar{x}$.

The rigid motion $\psi : E \to E$ can be lifted to the bundle of reference frames as follows: $(o, \{\vec{e}_i\}) \to (\psi(o), \{\psi_*(\vec{e}_i)\})$. To this change of frame corresponds an affine change of coordinates given by

$$\begin{pmatrix} \bar{x}' \\ 1 \end{pmatrix} = \begin{pmatrix} O & \bar{b} \\ \bar{0}^\intercal & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \tag{1}$$

where O is a special orthogonal matrix. Suppose that we perform a change of reference frame to which corresponds a change of coordinates given by (rigid map)

$$\begin{pmatrix} U & \bar{a} \\ \bar{0}^\intercal & 1 \end{pmatrix}, \qquad U \in SO(3),$$

then in the new frame the original rigid motion gets represented by the coordinate transformation matrix

$$\begin{pmatrix} U & \bar{a} \\ \bar{0}^\intercal & 1 \end{pmatrix} \begin{pmatrix} O & \bar{b} \\ \bar{0}^\intercal & 1 \end{pmatrix} \begin{pmatrix} U & \bar{a} \\ \bar{0}^\intercal & 1 \end{pmatrix}^{-1}.$$

Chasles' theorem states that the new reference frame can be chosen in such a way that the rigid motion in the newly defined coordinates is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix},$$

where $b \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. In other words, the motion is a rotation about the first axis followed by a translation of b along the same axis (since these two operations commute their order is irrelevant). If $\theta \neq 0$, the constant $p \in \mathbb{R}$ such that $b = \frac{p}{2\pi}\theta$, is called *pitch*.

It should be noted that a π-rotation of the reference frame on the plane $\mathrm{Span}(e_1, e_2)$ changes the sign of both θ and b. This operation makes it possible to choose the sign of b or to impose $\theta \in [0, \pi]$. We shall impose $b > 0$ whenever $b \neq 0$, and $\theta \in [0, \pi]$ whenever $b = 0$.

Algebrically, Chasles' theorem states that every conjugacy class in the matrix group of maps given by Eq. (1) has a representative of the above simplified form.

Here we are also interested in the reconstruction of the original rigid motion starting from the conjugacy class and other geometric data. The key observation is that by suitably limiting their domains, the parameters θ and b can

be uniquely determined, as they turn out to be independent of the simplifying reference frame. In the same way, if $\theta \neq 0$ the first axis of the simplifying reference frame does not depend on the frame (this is the characteristic *axis of rotation*). Thus each rigid map determines invariants of algebraic type (conjugacy class) and of geometrical type. Once put together they allow us to fully recover the rigid motion. Table 4 summarizes the families of conjugacy classes, the relevant parameters and their domain, the interpretation, and the necessary geometric ingredients needed to recover the isometry given the conjugacy class (parameters).

For instance, line (c3) clarifies that we cannot recover the rigid motion from the only information that it is a translation (i.e. type (c3)) of module $b > 0$. We need an additional normalized vector belonging to V which defines the direction of the translation (indeed, in case (c3) the simplifying reference frame can be freely translated, thus there is no characteristic line but only a characteristic oriented direction). Similarly, if we know that the isometry is a composition of a rotation and a translation ($\theta \in (0, 2\pi), b > 0$), then we need an oriented line in order to recover the rigid motion. In the special case of a rotation of angle π with $b = 0$, the orientation of the line is not needed (indeed, the first axis of the simplifying reference frame can point in both directions of the line).

We shall not comment these characterizations further as similar considerations will be made for the relativistic generalization. We end the section commenting table 2 in which we lists the conjugacy classes and the characteristic geometric invariants needed to reconstruct the isometry in Euler's case. It is worth noting that if the direction and verse of the rotation are represented using a normalized vector $v \in V$ then, joining the angle θ and this geometric object into $\theta v \in V$, we can represent the Lie group with a ball of radius π, in which opposite points in the exterior spherical surface have been identified. This is a well know geometrical representation of the group of rotations. This construction shows that the conjugacy classes correspond to the spherical surfaces inside the ball, the origin (the conjugacy class of the identity), and the real projective plane of its surface (the conjugacy class of π-rotations).

2.1 Infinitesimal (Lie algebra) formulation and screw product

The rigid motions $\psi : E \to E$ form a Lie group R. Near the identity the exponential map is a diffeomorphism, thus there is some element v of the Lie algebra $\mathfrak{R}$ such that $\psi = \exp(vs)$ for $s = 1$. Every point $p \in E$ gives an orbit $s \to \exp(vs)(p)$ and hence determines a vector tangent at p which we denote $v(p)$. Conversely, $v(p)$ determines a one-parameter group of diffeomorphisms $\psi_s : E \to E$, and $\psi = \psi_1$. Thus we may identify the Lie algebra element v with the vector field (denoted in the same way) $v : E \to V$.

Suppose we have chosen a reference frame. The matrix transformation $\psi =$

$\exp(v\epsilon)$ for small ϵ induces the coordinate change

$$\begin{pmatrix} \bar{x}' \\ 1 \end{pmatrix} = [I + \epsilon \begin{pmatrix} \Omega & \bar{c} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}] \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \tag{2}$$

where $\Omega \in \mathfrak{so}(3)$, i.e. it is a antisymmetric matrix, while $\bar{c}$ is a 3-vector. Thus we can also identify $\mathfrak{R}$ with $\mathfrak{iso}(3)$, namely the space of matrices of the form $\begin{pmatrix} \Omega & \bar{c} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$. This Lie algebra isomorphism depends on the reference frame, as the matrix representing the infinitesimal transformation changes under the Ad map of $ISO(3)$ on $\mathfrak{iso}(3)$ for changes of frame.

Let us find the corresponding Lie algebra vector field. Let us consider a point q of coordinates $\bar{x}$ on the given starting frame $\{\vec{e}_i\}$. This point is sent to $\psi(q)$, where $\psi(q)$ is the point with the same coordinates $\bar{x}$ but in the image frame $(o, \{\vec{e}_i\}) \to (\psi(o), \{\psi_*(\vec{e}_i)\})$. This means that for the starting frame $\psi(q)$ has coordinates $\bar{y} = \bar{x} - (\Omega\bar{x} + \bar{c})\epsilon$. Thus the vector field $v : E \to V$ is

$$v = -(\Omega_{ij} x^j + c^i)\vec{e}_i.$$

We observe that a vector field satisfies the above equation for some $\Omega \in \mathfrak{so}(3)$ and $\bar{c}$, if and only if there is a vector $\vec{\omega} \in V$ such that for every $p, q \in E$

$$v(p) - v(q) = \vec{\omega} \times (p - q). \tag{3}$$

The previous equation is the *constitutive equation of screws* where a screw is nothing but a Lie algebra element of the group of rigid motions. It can be shown that if a vector field is a screw then $\vec{\omega}$ is uniquely determined. We call it the *screw resultant*. If $\vec{\omega} \neq \vec{0}$ there is also a characteristic line on E called *screw axis*, which is the locus at which $|v(p)|$ attains the minimum [25, 37].

As we mentioned, the orbits of the Ad-action on $\mathfrak{iso}(3)$ might admit particularly simple representatives. This action corresponds to frame changes, thus the choice of matrix representative corresponds to a convenient frame choice. In particular, we can obtain a simple representative choosing a frame with the origin on the screw axis and first base element $\vec{e}_1$ aligned with the axis. In this way it is easy to show that the representative takes the forms (Lc1) and (Lc2) given by table 3, respectively in case $\vec{\omega} = \vec{0}$ and in case $\vec{\omega} \neq \vec{0}$.

On the Lie algebra of the group of rigid motions it is possible to define an important Ad-invariant indefinite inner product called *screw product*. Given two screws $v_1, v_2 : E \to V$ we define

$$\langle v_1, v_2 \rangle := v_1(p) \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot v_2(p). \tag{4}$$

By using equation (3) it can be easily shown that the definition is well posed as the right-hand side is independent of p. The screw product is particularly important in rigid body dynamics were the kinetic energy and the power action on a rigid body can be expressed through it [25, 37]. Contrary to a possible naive expectation, the screw product differs from the Killing form of the Lie algebra [25] (which is instead proportional to $\vec{\omega}_1 \cdot \vec{\omega}_2$, namely the scalar product of the resultants).

In a given reference frame the screw is determined by the pair $(\Omega,\ \bar{c})$. A calculation at the origin of the reference frame shows that the screw product is given by

$$\langle v_1, v_2 \rangle = \frac{1}{2}[\epsilon_{ijk}\Omega^{(1)}_{ij}c^{(2)}_k + \epsilon_{ijk}\Omega^{(2)}_{ij}c^{(1)}_k]. \tag{5}$$

It is clear that this expression is invariant under rotations of the frame. Under translations the Ω-s are left invariant while the $\bar{c}$ terms change as follows $c^{(2)}_k \to -\Omega^{(2)}_{kj}b_j + c^{(2)}_k$, $c^{(1)}_k \to -\Omega^{(1)}_{kj}b_j + c^{(1)}_k$. The additional terms cancel out, hence the screw product is Ad-invariant (for a different proof see [25]). In section 4.4 we shall meet a kind of relativistic generalization of the invariant (5).

Table 1: Euler's theorem and reconstruction (Lie algebra version)

Type	Families of orbits	Parameters	Description	Geometric ingredients
(Le1)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix}$	$\theta \neq 0$	rotation field	direction and verse

Table 2: Euler's theorem and reconstruction (Group version)

Type	Families of conjugacy classes	Parameters	Description	Geometric ingredients
(e1)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$	$\theta \in (0, \pi)$	rotation	direction and verse
(e2)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	[none]	π-rotation	direction

Table 3: Chasles' theorem and reconstruction (Lie algebra version)

Type	Families of orbits	Parameters	Description	Geometric ingredients
(Lc1)	$\begin{pmatrix} 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$b > 0$	translation field	direction and verse
(Lc2)	$\begin{pmatrix} 0 & 0 & 0 & -b \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\theta \neq 0$	screw field	oriented line

Table 4: Chasles' theorem and reconstruction (Group version)

Type	Families of conjugacy classes	Parameters	Description	Geometric ingredients
(c1)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\theta \in (0, \pi)$	rotation	oriented line
(c2)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	[none]	π-rotation	line
(c3)	$\begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$b > 0$	translation	direction and verse
(c4)	$\begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$b > 0$, $\theta \in (0, 2\pi)$	screw	oriented line

3 The Lorentz group

Let M be Minkowski spacetime, namely an affine space modeled over the vector space W, where $(W, \eta, or, \uparrow)$ is a 4-dimensional vector space endowed with an inner product η of signature $(-, +, +, +)$, an orientation or, and a time orientation $\uparrow$ (namely a choice of future and hence past timelike cone). A vector in the future cone will be called *future directed*, f.d. for short. The *proper orthochronous Lorentz group* $L_+^\uparrow$ is given by the set of automorphisms of W which respect both the orientation and the time orientation (an automorphism respects the time orientation if it sends the future timelike cone into itself). The inhomogeneous proper orthochronous Lorentz group $IL_+^\uparrow$ is made by the maps $P : M \to M$, which preserve the inner product η, the orientation, and which respect the time orientation. It can be shown (this fact can also be deduced from Alexandrov and Zeeman's theorem [2, 3, 3, 46] on causal automorphisms) that they are affine maps, namely they satisfy $P(p + w) = P(p) + \Lambda(w)$, for every $p \in M$, $w \in W$, where $\Lambda \in L_+^\uparrow$.

A *proper orthochronous orthonormal base* for W, is a positively oriented tetrad $\{e_a; a = 0, 1, 2, 3\}$, such that e_0 is timelike future directed and $\eta(e_a, e_b) = \eta_{ab}$ where $\eta_{00} = -1$, $\eta_{ii} = 1$, $i = 1, 2, 3$, and the other values vanish. Sometimes we shall refer to these bases as *reference frames*. Once a reference frame has been chosen, any vector $w \in W$ can be written as $w = w^a e_a$ for some components $w^a \in \mathbb{R}$, $a = 0, 1, 2, 3$.

Let $e'_a = \Lambda(e_a)$, then $\{e'_a\}$ is also a frame which can be expressed in terms of the old base as $e'_a = (\Lambda^{-1})^b{}_a e_b$. The change of reference frame induces a change in the components of a vector $w \in W$ as follows $w^{a\prime} = \Lambda^a{}_b w^b$. The choice of proper orthochronous orthonormal base establishes an isomorphism between the Lorentz group $L_+^\uparrow$ and the matrix proper orthochronous Lorentz group $SO(1, 3)^\uparrow$ given by the 4×4 matrices $\Lambda^a{}_b$ such that $\eta_{cd} = \eta_{ab}\Lambda^a{}_c\Lambda^b{}_d$, $\det(\Lambda^a{}_b) = 1$ and $\Lambda^0{}_0 > 0$. Let us focus on the action of Λ on a different frame $\tilde{e}_d = (L^{-1})^c{}_d e_c$. Let $\tilde{e}'_d = \Lambda(\tilde{e}_d) = (\tilde{\Lambda}^{-1})^c{}_d \tilde{e}_c$, then $\tilde{\Lambda}^c{}_d = L^c{}_a \Lambda^a{}_b (L^{-1})^b{}_d$. Thus, a change of frame acts as an automorphism $g \to cgc^{-1}$ of $SO(1, 3)^\uparrow$.

3.1 The Lie algebra and its orbits

The Lie algebra of the proper orthochronous Lorentz group $L_+^\uparrow$ is given by the skew-symmetric linear maps $F : W \to W$, that is by those maps such that, for every $w, v \in W$, $\eta(v, Fw) + \eta(Fv, w) = 0$. Any reference frame establishes a Lie algebra isomorphism between this Lie algebra and the Lie algebra $\mathfrak{so}(1, 3)$ of the matrix group $SO(1, 3)^\uparrow$ ($\mathfrak{so}(1, 3)^\uparrow$ and $\mathfrak{so}(1, 3)$ coincide because $SO(1, 3)^\uparrow$ is the connected component of $O(1, 3)$ which contains the identity)). As it is well known, $F^a_b \in \mathfrak{so}(1, 3)$ iff it is antisymmetric, $F_{ab} + F_{ba} = 0$, where the indices are lowered using η_{cd}.

The Ad-action of $SO(1, 3)^\uparrow$ on $\mathfrak{so}(1, 3)$ is given by $F \to LFL^{-1}$. When L runs over $SO(1, 3)^\uparrow$ we get an orbit of the Ad action on the Lie algebra. Each conjugacy transformation represents a change of frame, thus by looking at a

convenient representative in the orbit we are looking at frame which simplifies the matrix expression of the infinitesimal transformation.

The next result has long been established especially in connection with electromagnetism (where F represents an electromagnetic field). It can be regarded as a relativistic infinitesimal (i.e. Lie algebra) version of Euler's theorem.

Theorem 3.1. *Let $F: W \to W$ be a skew-symmetric linear map, then it is possible to choose a proper orthochronous orthornormal base $\{e_a\}$ such that the endomorphism F takes one of the following matrix forms*

$$(a) \quad A = \begin{pmatrix} 0 & -\varphi & 0 & 0 \\ -\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}; \qquad (b) \quad B = \begin{pmatrix} 0 & 0 & -\alpha & 0 \\ 0 & 0 & -\alpha & 0 \\ -\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\varphi > 0$, $\theta \in \mathbb{R}$, or $\varphi = 0$, $\theta \geq 0$ and where $\alpha \in \mathbb{R}$ can be chosen at will provided $\alpha \neq 0$. Stated in another way, the orbits of $\mathfrak{so}(1,3)$ under the Ad action of $SO(1,3)^{\uparrow}$ admit one and only one of the representatives given above (apart for the mentioned freedom in α) (the trivial orbit of the origin contains only the zero matrix).

Defined the invariants

$$I_1 = \frac{1}{4} F_{ab} F^{ab} = -\frac{1}{4} TrF^2,$$
$$I_2 = -\frac{1}{4} \epsilon_{abcd} F^{ab} F^{cd},$$

where $\epsilon_{0123} = 1$, we have $I_1 = (\theta^2 - \varphi^2)/2$, $I_2 = \theta\varphi$, thus it is possible to read the orbit calculating

$$\varphi = \sqrt{-I_1 + \sqrt{I_1^2 + I_2^2}}, \tag{6}$$

$$\theta = sgn(I_2) \sqrt{I_1 + \sqrt{I_1^2 + I_2^2}}, \tag{7}$$

(where $sgn(0) = 1$) provided φ or θ is different from zero (i.e. if we happen to be in case (a) where at least one of the invariant does not vanish). The map F is non-singular if and only if $I_2 \neq 0$.

Proof. A proof of the first claim can be found in [29, Sect. 2.4], [43, Sect. 9.5], [42, Sect. 9.3] or [26]. The latter claims follow easily. We give here a simple proof of the first claim. We start choosing any base. The matrix of the endomorphism F takes the form $\begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & b_3 & -b_2 \\ c_2 & -b_3 & 0 & b_1 \\ c_3 & b_2 & -b_1 & 0 \end{pmatrix}$. Under rotations of the reference frame the triples $\vec{c} = (c_1, c_2, c_3)$ and $\vec{b} = (b_1, b_2, b_3)$ transform as vectors. The invariants read $I_1 = \frac{1}{2}(\vec{b}^2 - \vec{c}^2)$, $I_2 = -\vec{c} \cdot \vec{b}$. We can choose the frame in such a way that $\vec{c} \propto e_2$, $\vec{c}, \vec{b} \in \mathrm{Span}(e_2, e_3)$, and the first axis is oriented

in the direction of $\vec{c}\times\vec{b}$ (so that $\vec{c}\times\vec{b}=(c_2b_3,0,0)$ where $c_2b_3\geq 0$). This choice simplifies the matrix because $c_1=c_3=b_1=0$. Furthermore, if the invariants vanish then $\vec{c}$ and $\vec{b}$ are perpendicular and of the same magnitude, thus we can choose $\vec{b}$ aligned with the third axis, and hence obtain (b). If $\vec{b}\propto\vec{c}$ then we obtain (a) aligning e_1 with them. In the remaining case $c_2b_3=|\vec{c}\times\vec{b}|>0$. Now, we make a boost in direction e_1 so that the endomorphism gets represented by the matrix

$$\begin{pmatrix} \cosh\gamma & -\sinh\gamma & 0 & 0\\ -\sinh\gamma & \cosh\gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 0 & c_2 & 0\\ 0 & 0 & b_3 & -b_2\\ c_2 & -b_3 & 0 & 0\\ 0 & b_2 & 0 & 0\end{pmatrix}\begin{pmatrix} \cosh\gamma & \sinh\gamma & 0 & 0\\ \sinh\gamma & \cosh\gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

so that $c_2'=c_2\cosh\gamma-b_3\sinh\gamma$, $b_3'=-c_2\sinh\gamma+b_3\cosh\gamma$, $c_3'=b_2\sinh\gamma$, $b_2'=b_2\cosh\gamma$ and $c_1'=b_1'=0$. We ask if we can find a value of γ which aligns $\vec{c}'$ with $\vec{b}'$. They are aligned if $\vec{c}'\times\vec{b}'=\vec{0}$ which holds if the next expression vanishes

$$c_2'b_3'-c_3'b_2'=-c_2^2\sinh^2\gamma-(b_2^2+b_3^2)\sinh\gamma\cosh\gamma+c_2b_3\cosh(2\gamma).$$

For $\gamma=0$ the right-hand side gives $c_2b_3>0$, while for large γ it goes as $\sim(-\vec{c}^2-\vec{b}^2+2|\vec{c}\times\vec{b}|)e^{2\gamma}/4$. Thus if $\vec{c}^2+\vec{b}^2>2|\vec{c}\times\vec{b}|$ it vanishes for some γ. This is the case because

$$(\vec{c}^2+\vec{b}^2)^2-(2|\vec{c}\times\vec{b}|)^2=(\vec{c}^2-\vec{b}^2)^2+4(\vec{c}\cdot\vec{b})^2=4(I_1^2+I_2^2)>0.$$

$\square$

The following identity shows that, indeed, the orbit of $B(\alpha)$, contains all the matrices of type $B(\alpha')$, with $\alpha'\neq 0$ (in order to change sign make a π-rotation of the frame on the plane $\mathrm{Span}(e_2,e_3)$)

$$\begin{pmatrix} \cosh\gamma & -\sinh\gamma & 0 & 0\\ -\sinh\gamma & \cosh\gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 0 & -\alpha & 0\\ 0 & 0 & -\alpha & 0\\ -\alpha & \alpha & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} \cosh\gamma & \sinh\gamma & 0 & 0\\ \sinh\gamma & \cosh\gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix} 0 & 0 & -\alpha' & 0\\ 0 & 0 & -\alpha' & 0\\ -\alpha' & \alpha' & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad \text{with } \alpha'=\alpha e^{-\gamma}. \tag{8}$$

Remark 3.2. The orbit with representative B cannot be distinguished from the trivial orbit using continuous invariant functions. Indeed, if $[B]$ is the orbit of B, then $\overline{[B]}$ contains the identity (take the limit $\alpha\to 0$ of $B(\alpha)$) and hence the function would take the same value on both orbits.

Remark 3.3. The physical content of the previous theorem is quite interesting. It tells us that any infinitesimal Lorentz transformation can be regarded as the frame dragging of points attached to a frame which is accelerating and rotating in two canonical ways. One with the acceleration and angular velocities aligned

(we can choose the first axis with a suitable rotation), and the other with acceleration and angular velocities which are perpendicular and of equal module. It also tells us that in the former case the modules of the acceleration and of the angular velocity do not depend on the frame that accomplishes the simplification, and thus can be regarded as genuine characteristics of the infinitesimal Lorentz transformation. In the latter case, on the contrary, the (equal) modules are not uniquely determined because they depend on the simplifying frame. Indeed, they change boosting the simplifying frame along the direction determined by the vector product between acceleration and angular velocity.

While this interpretation is correct, it should be kept in mind that Lorentz transformations act on W, not on M. As we shall see, the introduction of translations will allow us to assign, for fixed movement duration, a meaningful module to the acceleration and angular velocity, even in those cases in which they are perpendicular.

Remark 3.4. We have the following identities

$$e^A = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad e^B = \begin{pmatrix} 1+\alpha^2/2 & -\alpha^2/2 & -\alpha & 0 \\ \alpha^2/2 & 1-\alpha^2/2 & -\alpha & 0 \\ -\alpha & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is interesting to note that $\exp A$ preserves the null directions $e_0 \pm e_1$, while $\exp B$ leaves invariant the null vector $e_0 + e_1$.

Proposition 3.5. *Let $\Lambda \in SO(1,3)^{\uparrow}$ and $F \in \mathfrak{so}(1,3)$ be such that $\Lambda = e^F$, then Λ and F have the same eigenvectors.*

Proof. Of course, it is trivial that the eigenvectors for F are eigenvectors for Λ. The non-trivial direction is the opposite. It is easy to check that the claim holds for $F = A$ or $F = B$. Since by a conjugacy transformation we can always reduce the problem to this case, the claim holds in general. □

A Lie group G with a surjective exponential map $\exp : \mathfrak{g} \to G$ is called *exponential*. The Lorentzian generalization of Euler's and Chasles' theorems can be obtained from their infinitesimal versions thanks to the following result.

Theorem 3.6. *The exponential map $\exp : \mathfrak{so}(1,3) \to SO(1,3)^{\uparrow}$ is surjective.*

Exponential Lie groups are very much studied in the literature [11] and the previous result is well established [34, 40, 31, 11, 27] [17, Theor. 6.5] [16, Theor. 4.21], see also [9, 8]. The nice fact is that although $SL(2,\mathbb{C})$ provides a double covering of $SO(1,3)^{\uparrow}$, the exponential $\exp : \mathfrak{sl}(2,\mathbb{C}) \to SL(2,\mathbb{C})$ is not surjective [27, 16] (the group $SL(2,\mathbb{R})$ is often used to shown that the exponential does not need to be surjective [12]). Indeed, the matrix

$$\begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix}, \qquad h \neq 0$$

does not belong to any 1-parameter subgroup of $SL(2,\mathbb{C})$.

3.2 Lorentzian extension of Euler's theorem

We formulate the Lorentzian generalization of Euler's theorem.

Theorem 3.7. *Let $\Lambda : W \to W$ be a non-trivial proper orthochronous Lorentz transformation, then we can find a proper orthochronous orthonormal base in such a way that the matrix ${\Lambda^a}_b$ belongs to the 2-dimensional Abelian subgroup of roto–boosts*

$$(a): \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \qquad \text{with: } \begin{array}{l} \varphi > 0,\ \theta \in [0, 2\pi),\ \text{or} \\ \varphi = 0,\ \theta \in [0, \pi] \end{array}$$

or to the 1-dimensional Abelian subgroup of null (Galileian) boosts

$$(b): \begin{pmatrix} 1+\alpha^2/2 & -\alpha^2/2 & -\alpha & 0 \\ \alpha^2/2 & 1-\alpha^2/2 & -\alpha & 0 \\ -\alpha & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

If (b) applies with $\alpha \neq 0$ then α can be chosen arbitrarily (as long as it is different from zero). Apart from this freedom, which does not change the conjugacy class, different matrices correspond to different conjugacy classes.

The matrix is of type (a) if and only if $\Lambda : W \to W$ leaves invariant a timelike 2-subspace, if and only if $\Lambda : W \to W$ leaves invariant at least two lightlike directions. The matrix is of type (a) with $\theta = 0$, if and only if it is of type (a) and leaves invariant one spacelike vector (and hence every vector in a spacelike 2-subspace). The matrix is of type (a) with $\varphi = 0$, if and only if it is of type (a) and leaves invariant at least two lightlike vectors. The matrix is of type (b) with $\alpha \neq 0$ if and only if $\Lambda : W \to W$ leaves invariant one and only one lightlike vector.

More specifically, the reference frame can be chosen in such a way that the matrix takes one and only one of the forms given in table 6. The type and the parameters' value are independent of the simplifying reference frame and, moreover, the simplifying reference frame fixes unambiguously some geometric data given in the last column of the table. Furthermore, if the type, the parameters and the geometric data are given, then the transformation can be completely determined.

Proof. By theorem 3.6 there is some antisymmetric matrix F such that $\Lambda = \exp F$. We choose the proper orthochronous orthonormal base in such a way that F takes one of the canonical forms given by theorem 3.1. Thus by suitably choosing the proper orthochronous orthonormal base we can make Λ to take the form $\exp A$ or $\exp B$ given by remark 3.4. This is the first claim of the theorem. From here the other statements follow with little effort. □

Transformations of type (b) might be called *Galileian boosts*. The justification of this terminology can be found in [24], where it is shown that they provide a 1-dimensional subgroup of the group $E(2)$ of Galileian boost in 2-dimensions plus rotations (see also [13, 14]).

Table 5: Relativistic Euler's theorem and reconstruction (Lie algebra version)

Type	Families of orbits	Parameters	Description	Geometric ingredients
(Ll1)	$\begin{pmatrix} 0 & -\varphi & 0 & 0 \\ -\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}$	$\varphi = 0$, $\theta > 0$, or $\varphi > 0$	roto–boost field	oriented timelike 2-subspace
(Ll2)	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	[none]	null (Galileian) boost field	oriented lightlike 2-subspace and f.d. lightlike vector on it.

Table 6: Relativistic Euler's theorem and reconstruction (Group version)

Type	Families of conjugacy classes	Parameters	Description	Geometric ingredients
(l1)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$	$\theta \in (0, \pi)$	rotation	oriented timelike 2-subspace
(l2)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	[none]	π-rotation	timelike 2-subspace
(l3)	$\begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$	$\varphi > 0$, $\theta \in [0, 2\pi)$	roto–boost	oriented timelike 2-subspace
(l4)	$\begin{pmatrix} 1+\alpha^2/2 & -\alpha^2/2 & -\alpha & 0 \\ \alpha^2/2 & 1-\alpha^2/2 & -\alpha & 0 \\ -\alpha & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	Any $\alpha \neq 0$ gives the same conjugacy class	null (Galileian) boost	oriented lightlike 2-subspace and f.d. lightlike vector on it.

Remark 3.8. Let us clarify the role of the geometric data.

Suppose that (a) applies with $\varphi = 0$, $\theta \neq 0$. Choose a simplifying frame such that $\theta \in [0, \pi]$. The invariant timelike 2-dimensional subspace $\mathrm{Span}(e_0, e_1)$ selected in this way is independent of the simplifying reference frame (precisely because it is characterized as timelike invariant subspace of $\Lambda : W \to W$). Furthermore, if $\theta \neq \pi$ the orientation of this subspace given by (e_0, e_1) is independent of the simplifying reference frame.

Suppose that (a) applies with $\varphi \neq 0$. Choose a simplifying frame such that $\varphi > 0$, and assign to the invariant timelike 2-dimensional subspace $\mathrm{Span}(e_0, e_1)$ the orientation given by the base (e_0, e_1). The invariant oriented timelike 2-subspace selected in this way is independent of the simplifying reference frame.

Suppose that (b) applies with $\alpha \neq 0$. Choose a simplifying frame such that $\alpha = 1$, and assign to the invariant lightlike 2-subspace $\mathrm{Span}(e_0 + e_1, e_3)$ the orientation given by the base $(e_0 + e_1, e_3)$. The invariant oriented lightlike 2-subspace selected in this way and the lightlike vector $e_0 + e_1$ are independent of the simplifying reference frame (for, another simplifying frame would be related to the former by a little group transformation of the vector $(1, 1, 0, 0)$. From here, since a null 2-plane must be left invariant, the frame change matrix must actually be a Galileian boost [24] with direction in $\mathrm{Span}(e_2, e_3)$.).

The map Λ can be completely recovered knowing, to start with, if it is of type (a) or (b). In case (a) it is sufficient to know the invariant oriented timelike 2-subspace and the constants $|\varphi|$, θ. In case (b) it is sufficient to know the invariant oriented lightlike 2-subspace which admits a base of invariant vectors, and the distinguished future directed lightlike vector on it.

The many paragraphs of the theorem serve to clarify the qualitative features of the Lorentz transformations. Two transformation which differ by these aspects cannot be related by conjugacy (for other characterizations see [39] [17, Theor. 6.1]).

We mention here another interesting approach to the study of conjugacy classes. It uses the isomorphism between the Lorentz group and $PSL(2, \mathbb{C})$. The idea comes from the observations that Lorentz transformations of the observer induce an action of $PSL(2, \mathbb{C})$ on the Riemann sphere of the 'night sky' [32, 33, 29].

According to the classification of conjugacy classes for the $PSL(2, \mathbb{C})$ group [30], the classes corresponding to matrices of the form (a) with $\theta = 0$, $\varphi \neq 0$, are called *hyperbolic*, those corresponding to matrices of the form (a) with $\theta \neq 0$, $\varphi = 0$, are called *elliptic*, those corresponding to matrices of the form (a) with $\theta \neq 0$, $\varphi \neq 0$, are called *loxodromic*, and that corresponding to matrices of the form (b) with $\alpha \neq 0$, is called *parabolic*. The class of the identity contains only the identity and is referred as the trivial class. We shall extend this terminology to the Lorentz transformations themselves. Thus a Lorentz transformation is hyperbolic if its conjugacy class is hyperbolic.

Since the parameters φ and θ are expressible in terms of Ad-invariants, matrices obtained for distinct parameters correspond to distinct conjugacy classes (up to the remarked freedom in the parameters choice).

Let us instead show that the parameters have the mentioned freedom. Suppose we are in case (a) with $\varphi = 0$. A π-rotation of the reference frame in the plane $\mathrm{Span}(e_1, e_2)$, changes the sign of θ which is then redefined adding 2π. As a result θ can be changed to take value in $[0, \pi]$.

Suppose we are in case (a) with $\varphi \neq 0$. In order to show that only the sign of φ is relevant for the conjugacy class, it is again sufficient to perform a π-rotation of the reference frame in the plane $\mathrm{Span}(e_1, e_2)$, as it changes the signs of both φ and θ which is then redefined adding 2π.

In case (b) different modules for α do not give different conjugacy classes (if $\alpha = 0$ we get the class of the identity), because of the identity

$$\begin{pmatrix} \cosh\gamma & -\sinh\gamma & 0 & 0 \\ -\sinh\gamma & \cosh\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\alpha^2/2 & -\alpha^2/2 & -\alpha & 0 \\ \alpha^2/2 & 1-\alpha^2/2 & -\alpha & 0 \\ -\alpha & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\gamma & \sinh\gamma & 0 & 0 \\ \sinh\gamma & \cosh\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\alpha'^2/2 & -\alpha'^2/2 & -\alpha' & 0 \\ \alpha'^2/2 & 1-\alpha'^2/2 & -\alpha' & 0 \\ -\alpha' & \alpha' & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } \alpha' = \alpha e^{-\gamma}. \tag{9}$$

The sign of α is also irrelevant because a π-rotation of the reference frame on the plane $\mathrm{Span}(e_2, e_3)$ replaces α with $\alpha' = -\alpha$. Thus, there will be a reference frame for which $\alpha = 1$. The lightlike vector $e_0 + e_1$ in this frame is then rather special (what is special of it is of course its normalizing module), and it is the invariant future directed lightlike vector to which the reconstruction statement of the theorem refers to.

Let us justify the reconstruction claim of the theorem. Suppose, for instance, that we are given the invariant timelike oriented 2-subspace and constants $\varphi > 0$, $\theta \in [0, \pi]$. We choose a vector e_0, timelike, normalized and future directed on the plane and e_1 orthogonal to it in such a way that (e_0, e_1) is positively oriented. Next we choose (e_2, e_3) spacelike, normalized, and orthogonal among themselves and with respect to the distinguished timelike plane. We order them in such a way that (e_0, e_1, e_2, e_3) is positively oriented. The linear map $\Lambda : W \to W$ whose matrix form in the base $\{e_a\}$ is given by matrix (a) is well defined as it is independent of the chosen base. The independence comes from the fact that the matrix form (a) is invariant under boosts of the frame in the e_1 direction and under rotations on the spacelike plane $\mathrm{Span}(e_2, e_3)$.

The first part of theorem 3.7 is essentially known [45, 1, 23, 35, 36, 39]. Sometimes the conjugacy classes are incorrectly identified, either confusing the family of classes as one single conjugacy class, or not realizing that the matrices of type (b) with $\alpha \neq 0$ belong to the same conjugacy class (we stress that there is only one parabolic conjugacy class).

The correct identification of the conjugacy classes is important but this data does not allow us to recover the transformation Λ. Two transformations belong to the same conjugacy class if it is possible to find two observers (proper orthochronous orthonormal bases) with respect to which they look the same [39].

The last sentence of the theorem allows us to extract the true physical content of the transformation Λ. It is important to realize that in case (a) with $\varphi \neq 0$, the timelike oriented 2-subspace and the coefficients $|\varphi|$, θ, have physical significance as they are independent of the reference frame. In the same way it is important to realize that in case (b), α has no physical significance while the oriented lightlike 2-subspace and the normalizing lightlike vector on it, do have. Thus the same parabolic conjugacy class corresponds to different parabolic Lorentz transformations Λ, as there are many distinct pairs made by an oriented lightlike 2-subspace and a future directed lightlike vector on it.

This analysis shows that the rotation axis of Euler's theorem is replaced here by an oriented causal plane passing through the origin in which the future direction is suitably normalized (this normalization can be omitted in the timelike case given the existence of a Lorentzian induced metric). The rotation angle in Euler's theorem is instead replaced by parameters $|\varphi|$, θ (in case (a)).

Given a Lorentz transformation $\Lambda : W \to W$ it is possible to read its conjugacy class through its characteristic polynomial $p(\lambda) = \det(\Lambda - \lambda I)$.

Theorem 3.9. *Let $\Lambda \in L^{\uparrow}_{+}$, $\Lambda \neq I$, then the characteristic polynomial reads*

$$
\begin{aligned}
(a') \qquad & p(\lambda) = (\lambda^2 - 2\cos\theta\,\lambda + 1)(\lambda^2 - 2\cosh\varphi\,\lambda + 1), \\
(b') \qquad & p(\lambda) = (\lambda - 1)^4,
\end{aligned}
$$

where (a') holds iff case (a) of theorem 3.7 applies, and (b') holds iff case (b) of theorem 3.7 applies. In particular, it is possible to distinguish between cases (a) and (b) and, if case (a) applies, to read $\varphi \in [0, +\infty)$, and $\theta \in [0, 2\pi)$. Indeed, let $p(\lambda) = \lambda^4 - p_3\lambda^3 + p_2\lambda^2 - p_1\lambda + 1$, then

$$
\begin{aligned}
p_3 &= p_1 = 2(\cosh\varphi + \cos\theta), \\
p_2 &= 2 - 4\cosh\varphi\,\cos\theta,
\end{aligned}
$$

hence

$$
\begin{aligned}
\cosh\varphi &= \frac{1}{4}[p_1 + \sqrt{p_1^2 + 4p_2 - 8}], \\
\cos\theta &= \frac{1}{4}[p_1 - \sqrt{p_1^2 + 4p_2 - 8}].
\end{aligned}
$$

Under the above assumption, namely $\Lambda \neq I$, the result $\theta = \varphi = 0$ implies that case (b) applies.

Proof. It is sufficient to calculate the characteristic polynomials for cases (a) and (b) of theorem 3.7, and to check the algebra. □

In case (b) all values $\alpha \neq 0$ correspond to the same conjugacy class. For $\alpha \to 0$ the matrix representatives converge to the identity, which belongs to a different conjugacy class. As a consequence, the conjugacy class given by (b) is not topologically closed in the topology of the Lie group. Thus, it is impossible to distinguish between $\Lambda = I$, and case (b) with $\Lambda \neq I$, by looking at the Ad-invariant continuous functions of Λ.

The conjugacy classes of type (a) are topologically closed. Indeed, the coefficients of the characteristic polynomial $p_1(\Lambda), p_2(\Lambda)$, are polynomials in the matrix coefficients of Λ, and hence are continuous in the Lie group topology. Functions $\varphi(\Lambda), \theta(\Lambda)$, being continuous in p_1, p_2, are also continuous with respect to the Lie group topology. Each conjugacy class is determined by its value $(\varphi, \theta) \in B := [0, +\infty) \times [0, 2\pi)$. The inverse image of a B point (which is closed) through the continuous map $(\varphi \times \theta)(\Lambda)$ is a closed set, hence conjugacy classes of type (a) are closed. Finally, the conjugacy class of the identity is closed because it is just a point in the Lie group. In summary.

Proposition 3.10. *The only conjugacy class of the Lorentz group which is not topologically closed is that of type (b). The closure of this class contains the identity.*

Through this same argument we can prove something more. Observe that function $(\varphi \times \theta)(\Lambda)$ is invariant under conjugation, thus so are the open (closed) sets obtained as inverse images of open (closed) sets. In particular, every distinct pair of conjugacy classes of type (a) is separated by invariant open sets.

4 The inhomogeneous Lorentz group

When working on the affine space M, by *reference frame* we shall mean an ordered pair $(o, \{e_a\})$ given by an *origin* $o \in M$ and a proper orthochronous orthonormal base $\{e_a\}$. A reference frame is then a point in the $SO(1,3)^{\uparrow}$-bundle R of reference frames [22]. Once a reference frame has been chosen, any point $p \in M$ can be written in a unique way in coordinates $\{x^a; a = 0, 1, 2, 3\}$, as $p = o + x^a e_a$. For short, from now on we shall denote the coordinate vector belonging to $\mathbb{R}^4$ using a bar, e.g. $\bar{x}$.

As we mentioned in section 3, a map $P \in IL_+^{\uparrow}$ satisfies $P(p + w) = P(p) + \Lambda(w)$, for every $p \in M$ and $w \in W$, where $\Lambda \in L_+^{\uparrow}$. The map P lifts to the bundle of reference frames as follows

$$(o, \{e_a\}) \xrightarrow{P} (P(o), \{e'_a\}), \quad \text{where } e'_a = \Lambda(e_a) = (\Lambda^{-1})^b{}_a e_b.$$

Once a reference frame has been chosen, an inhomogeneous proper orthochronous Lorentz transformation P induces a change of coordinates

$$x^{a\prime} = \Lambda^a{}_b x^b - b^a,$$

where Λ^a_b belongs to $SO(1,3)^{\uparrow}$ and $P(o) - o = (\Lambda^{-1})^c{}_d b^d e_a$. We shall write this coordinate transformation as

$$\begin{pmatrix} \bar{x}' \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}. \tag{10}$$

These matrices form the group $ISO(1,3)^{\uparrow}$. The choice of reference frame establishes an isomorphism between the inhomogeneous Lorentz group $IL_+^{\uparrow}$

and $ISO(1,3)^{\uparrow}$. A change of frame acts as an automorphism $g \to cgc^{-1}$ of $ISO(1,3)^{\uparrow}$. We remark that the equation $P(o) - o = (\Lambda^{-1})^c{}_d b^d e_a$ shows that $P(o)$ is in the causal (chronological) future of o if and only if b^d is future directed and causal (resp. timelike).

4.1 The causal semigroup of $ISO(1,3)^{\uparrow}$

The product of two elements of $ISO(1,3)^{\uparrow}$ gives

$$\begin{pmatrix} \Lambda_2 & -\bar{b}_2 \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & -\bar{b}_1 \\ \bar{0}^{\intercal} & 1 \end{pmatrix} = \begin{pmatrix} \Lambda_2\Lambda_1 & -(\Lambda_2\bar{b}_1 + \bar{b}_2) \\ \bar{0}^{\intercal} & 1 \end{pmatrix}$$

as a consequence, if $\bar{b}_1$ and $\bar{b}_2$ are future directed and nonspacelike so is $\Lambda_2\bar{b}_1 + \bar{b}_2$. Thus we have a semigroup on $ISO(1,3)^{\uparrow}$ which we call the (future directed) *causal semigroup of* $ISO(1,3)^{\uparrow}$. We denote it by J. Analogous considerations hold for $\bar{b}$ f.d. and timelike, and correspondingly we have a timelike semigroup I. Clearly, $I \subset J$. Notice that J, contrary to I, contains the identity, thus it is a *monoid*. The set $J \cap J^{-1}$ is the largest group contained in J and it is isomorphic to $SO(1,3)^{\uparrow}$ because it is made by those matrices for which $\bar{b} = \bar{0}$.

The semigroups I and J are important for the following reason. Suppose that $P : M \to M$, $P \in IL_+^{\uparrow}$, is such that there is a reference frame for which its matrix expression belongs to I (resp. J). As we observed in section 3, this means that the origin o of the frame that realizes the matrix reduction is sent to $P(o)$ where $P(o) - o$ is f.d. timelike (resp. f.d. nonspacelike). At least in the timelike case we can interpret this transformation as physically admissible. Indeed, we can select a special point of space, namely the origin o, which moves forward in time along a timelike geodesic segment. The transformation P can then be interpreted as an active transformation induced by the dragging of spacetime points along with this frame. Furthermore, we known that the matrix expression changes by conjugacy under frame changes. Thus, in order to find if a transformation P falls into this admissible class we have to find if the conjugacy class of the matrix transformation, obtained in a generic frame, admits some representative which belongs to I. We summarize this result with the following proposition.

Proposition 4.1. *The map $P \in IL_+^{\uparrow}$ sends some point of M in its causal (chronological) future if and only if there is a representative in the conjugacy class of its matrix representation which belongs to the semigroup J (resp. I).*

Proof. Suppose that some point $o \in M$ is sent into its causal future. Choose a frame at o, then the matrix representation of P belongs to J. The other direction has been proved above. □

While the group $ISO(1,3)^{\uparrow}$ is the group of symmetries of the spacetime manifold, the semigroup I distinguishes itself as the semigroup of symmetries that can be induced by the actual physical movement of a frame on M. Here the elements $ISO(1,3)^{\uparrow}$ which have to be discarded are those for which there

is no point that is sent to its chronological future. The elements in $E := J\setminus I$ are rather special. These are maps which respect causality at some point but which do not represent the physical movement of a massive reference frame.

4.2 The Lie algebra and its interpretation

The choice of reference frame establishes an isomorphism between $IL_+^{\uparrow}$ and the group $ISO(1,3)^{\uparrow}$ made of matrices $\begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^T & 1 \end{pmatrix}$ where $\Lambda \in SO(1,3)^{\uparrow}$ and $\bar{b} \in \mathbb{R}^4$. Furthermore, it establishes a Lie algebra isomorphism between the Lie algebra of $IL_+^{\uparrow}$, $\mathfrak{IL}$, and the Lie algebra $\mathfrak{iso}(1,3)$ made of matrices $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^T & 0 \end{pmatrix}$, where $F \in \mathfrak{so}(1,3)^{\uparrow}$ and $\bar{w} \in \mathbb{R}^4$, and where the commutator in $\mathfrak{iso}(1,3)$ is the usual matrix commutator.

Let us remind that $F \in \mathfrak{so}(1,3)^{\uparrow}$ iff $F^a_{\ b}$ satisfies $F_{ab} + F_{ba} = 0$, where the indices are lowered using η_{cd}. This Lie algebra coincides with the Lie algebra of the group $O(1,3)$ (because $SO(1,3)^{\uparrow}$ is the connected component of $O(1,3)$ which contains the identity).

A significative base for $\mathfrak{iso}(1,3)$ is given by

$$\begin{aligned} J^{ab} &= \begin{pmatrix} M^{ab} & \bar{0} \\ \bar{0}^{\mathsf{T}} & 0 \end{pmatrix}, \\ P^a &= \begin{pmatrix} 0 & \bar{m}^a \\ \bar{0}^{\mathsf{T}} & 0 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} (M^{ab})^c_{\ d} &= \eta^{ac}\delta^b_d - \eta^{bc}\delta^a_d, \\ (\bar{m}^a)^c &= \eta^{ac}. \end{aligned}$$

The subalgebra generated by J^{ab} is the Lie algebra $\mathfrak{so}(1,3)$. The non-vanishing commutation relations are

$$[J^{ab}, J^{cd}] = \eta^{ad}J^{bc} + \eta^{bc}J^{cd} - \eta^{ac}J^{bd} - \eta^{bd}J^{ac}, \tag{11}$$

$$[J^{ab}, P^c] = \eta^{cb}P^a - \eta^{ca}P^b. \tag{12}$$

We shall write $H = P^0$. We introduce the generators

$$\begin{aligned} K^i &= J^{0i}, \\ J^i &= \frac{1}{2}\epsilon_{ijk}\, J^{jk}, \qquad (J^{jk} = \epsilon_{ijk}\, J^i). \end{aligned}$$

The non-vanishing commutation relations are (lowering space indices does not introduce minus signs)

$$\begin{aligned} &[J_i, J_j] = -\epsilon_{ijk}\, J_k, \qquad [J_i, K_j] = -\epsilon_{ijk}\, K_k, \qquad [K_i, K_j] = \epsilon_{ijk}\, J_k, \\ &[J_i, P_j] = -\epsilon_{ijk}\, P_k, \qquad [K_i, P_j] = \delta_{ij}\, H, \qquad [K_i, H] = P_i. \end{aligned}$$

The following matrix expressions clarify our conventions (which are the same of [24]).

$$K^1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Arguing as in section 2.1 we find that $\mathfrak{IL}$ is a subalgebra of the algebra of vector fields on M, the correspondence between matrices and vector fields being ($e_d = \partial_d$)

$$I + \epsilon[\frac{1}{2}\Omega_{ab}J^{ab} - b_c P^c] \xleftrightarrow{\{e_a\}} [-\Omega^d{}_b x^b + b^d]\, e_d, \tag{13}$$

By *observer* we mean a f.d. timelike worldline $\tau \to \gamma(\tau)$, parametrized with respect to proper time and a reference frame $\{e_a\}(\tau)$ over it such that at any point $e_0 = \partial_\tau$. If the tetrad is parallely transported the observer is *inertial*. Starting from $\{e_a\}(\tau_0)$ one can regard the motion of the observer as the repeated action of Lorentz transformations $\delta\Lambda \in IL_+^\uparrow$ sending $(\gamma(\tau), \{e_a\}(\tau))$ to $(\gamma(\tau+\delta\tau), \{e_a\}(\tau+\delta\tau))$. At each instant we have coordinates x_τ^a associated to the frame $\{e_a\}(\tau)$, thus $\delta\Lambda(\tau)$ induces a change of coordinates. Let $\vec{a}$ be the acceleration of the observer, and let $\vec{\omega}$ be its angular velocity (all quantities are measured by herself). The coordinate change induced by the motion of the observer in a proper time interval $\mathrm{d}\tau$ is

$$I + (a_i K^i + \omega_k J^k + H)\mathrm{d}\tau. \tag{14}$$

as it can be easily inferred from its matrix form (see [24] for another argument). In this equation $a_i = \Omega_{0i}$, and $\omega_k = \frac{1}{2}\epsilon_{kij}\Omega_{ij}$, where we have identified the small parameter ϵ with $\mathrm{d}\tau$. Thus the infinitesimal motion of the observer is given by matrix (13) for $b^0 = 1$ and $b^i = 0$ for $i = 1, 2, 3$. The vector field which generates the infinitesimal transformation of M, and hence the change of reference frame to which correspond the coordinate change (14), is

$$\partial_0 + a_i(x^0\partial_i + x^i\partial_0) + \omega_k(\epsilon_{kij}x^i\partial_j) \tag{15}$$

where we used the corresponences

$$K_i \leftrightarrow x^0\partial_i + x^i\partial_0, \qquad J_i \leftrightarrow \epsilon_{kij}x^i\partial_j,$$
$$H \leftrightarrow \partial_0, \qquad P_i \leftrightarrow -\partial_i.$$

It is easy to check that these vector fields satisfy the same commutation relations of their matrix counterparts.

It is natural to ask why in the full inhomogeneous Lorentz group we have to consider translations generated by P^i if they do not appear as Lie algebra generators of the observer's movements. The answer is that the operators P^i arise at the non-infinitesimal level, through the compositions of several operations of the above type, as a consequence of the Lie algebra commutation relations (e.g. through Baker-Campbell-Hausdorff formula).

4.3 Exponentiality of $ISO(1,3)^{\uparrow}$

Let us consider the equation which defines the exponential map

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\Lambda & -\bar{b}\\ \bar{0}^{\intercal} & 1\end{pmatrix}=\begin{pmatrix}F & -\bar{w}\\ \bar{0}^{\intercal} & 0\end{pmatrix}\begin{pmatrix}\Lambda & -\bar{b}\\ \bar{0}^{\intercal} & 1\end{pmatrix},$$

with initial condition $\Lambda = I$, $\bar{b} = \bar{0}$. The matrix equation is equivalent to the system

$$\frac{\mathrm{d}}{\mathrm{d}s}\Lambda = F\Lambda, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\bar{b} = F\bar{b} + \bar{w}. \tag{17}$$

Let $\bar{c}(s)$ be such that $\bar{b}(s) = \Lambda(s)\bar{c}(s)$ (thus $\bar{b}$ is f.d. nonspacelike iff $\bar{c}$ is). Equation (17) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}s}\bar{c}(s) = \Lambda^{-1}(s)\bar{w}. \tag{18}$$

Through these equations, and using the exponentiality of $SO(1,3)^{\uparrow}$, we are now able to prove.

Theorem 4.2. *The group $ISO(1,3)^{\uparrow}$ is exponential. More in detail, for every $F \in \mathfrak{so}(1,3)$ the matrix $(e^F - I)/F$ is well defined and invertible. Let $\begin{pmatrix}\Lambda & -\bar{b}\\ \bar{0}^{\intercal} & 1\end{pmatrix} \in ISO(1,3)^{\uparrow}$ (that is, $\Lambda \in SO(1,3)^{\uparrow}$), then there is some $F \in \mathfrak{so}(1,3)$ such that $\Lambda = \exp F$ and for any such choice the vector $\bar{w}$ defined by*

$$\bar{w} = \frac{F}{e^F - I}\,\bar{b}$$

is such that, $\exp[\begin{pmatrix}F & -\bar{w}\\ \bar{0}^{\intercal} & 0\end{pmatrix}] = \begin{pmatrix}\Lambda & -\bar{b}\\ \bar{0}^{\intercal} & 1\end{pmatrix}$ *and is the only vector with this property. Furthermore, $\bar{w}$ is a f.d. null eigenvector of F with eigenvalue $\lambda \in \mathbb{R}$, if and only if $\bar{b}$ is a f.d. null eigenvector of Λ with eigenvalue* $\exp\lambda \in (0,+\infty)$*, in which case $\bar{w} = \frac{\lambda}{e^{\lambda}-1}\bar{b}$.*

Proof. The function $f(z) = (e^z - 1)/z$ is analytic thus it makes sense to consider $f(F)$, for $F \in \mathfrak{so}(1,3)$. Since f satisfies $f(LFL^{-1}) = Lf(F)L^{-1}$ for $L \in SO(1,3)^{\uparrow}$, in order to prove its invertibility we have just to verify this property over one representative for each orbit on $\mathfrak{so}(1,3)$. Therefore, we can use the representatives A and B selected in theorem 3.1.

If $F = A$ then

$$(e^A - I)/A = \begin{pmatrix} \frac{\sinh\varphi}{\varphi} & \frac{1-\cosh\varphi}{\varphi} & 0 & 0 \\ \frac{1-\cosh\varphi}{\varphi} & \frac{\sinh\varphi}{\varphi} & 0 & 0 \\ 0 & 0 & \frac{\sin\theta}{\theta} & \frac{1-\cos\theta}{\theta} \\ 0 & 0 & \frac{1-\cos\theta}{\theta} & \frac{\sin\theta}{\theta} \end{pmatrix},$$

which has positive determinant $\frac{2(\cosh\varphi-1)}{\varphi^2}\frac{2(1-\cos\theta)}{\theta^2}$ (this expression makes sense and is finite for $\varphi = 0$ or $\theta = 0$).

If $F = B$ we can use $B^3 = 0$ so that

$$(e^B - I)/B = I + \frac{B}{2} + \frac{B^2}{6} = \begin{pmatrix} 1+\alpha^2/6 & -\alpha^2/6 & -\alpha/2 & 0 \\ \alpha^2/6 & 1-\alpha^2/6 & -\alpha/2 & 0 \\ -\alpha/2 & \alpha/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which has determinant equal to 1.

Let us try to find $\bar{w}$ in such a way that $\exp[\begin{pmatrix} F & -\bar{w} \\ \bar{0}^\intercal & 0 \end{pmatrix}] = \begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^\intercal & 1 \end{pmatrix}$. From Eq. (18) it follows

$$\bar{c}(1) = [\int_0^1 \Lambda^{-1}(s)\mathrm{d}s]\bar{w}$$

and we must comply with $\bar{b}(1) = \Lambda(1)\bar{c}(1)$. But the solution to Eq. (16) is $\Lambda(s) = \exp[Fs]$, and by assumption $\Lambda = \Lambda(1) = \exp F$, thus

$$\bar{b} = (\exp F)\bar{c}(1) = \exp F[\int_0^1 e^{-Fs}\mathrm{d}s]\bar{w} = \frac{\exp F - I}{F}\,\bar{w}$$

Since the matrix on the right-hand side is invertible there is one and only one vector $\bar{w}$ which complies with this equation.

Finally, the last statement follows easily from the fact that Λ and F have the same eigenvectors (Prop. 3.5).

□

4.4 Ad-invariants and Lie algebra orbits

Let us consider the Ad-action of $ISO(1,3)^\uparrow$ on $\mathfrak{iso}(1,3)$,

$$\begin{pmatrix} L & -\bar{a} \\ \bar{0}^\intercal & 1 \end{pmatrix} \begin{pmatrix} F & -\bar{w} \\ \bar{0}^\intercal & 0 \end{pmatrix} \begin{pmatrix} L & -\bar{a} \\ \bar{0}^\intercal & 1 \end{pmatrix}^{-1} = \begin{pmatrix} LFL^{-1} & LFL^{-1}\bar{a} - L\bar{w} \\ \bar{0}^\intercal & 0 \end{pmatrix}.$$

We shall be interested in the separated effect of

(i): homogeneous transformations of the frame

$$\begin{aligned} &F \to LFL^{-1}, \\ &\bar{w} \to L\bar{w}, \end{aligned} \tag{19}$$

(ii): translations of the frame

$$\begin{aligned} &F \to F, \\ &\bar{w} \to \bar{w} - F\bar{a}. \end{aligned} \tag{20}$$

The action on the homogeneous part F coincides with the Ad-action of $SO(1,3)^{\uparrow}$ on $\mathfrak{so}(1,3)$. Thus the invariants I_1 and I_2 of section 3.1 are still invariants for the Ad action on the inhomogeneous Lie algebra. We are going to show that if $I_2 = 0$ then there is a third invariant I_3. It represents a kind of relativistic generalization of the square of the screw scalar $\langle v, v\rangle$ (unfortunately, there does not seem to be any convenient relativistic generalization of the screw product).

It will be convenient to keep in mind that the most generic frame transformation can be accomplished through a translation followed by a homogeneous transformation, according to this scheme

$$\begin{aligned} &F \to F \to LFL^{-1}, \\ &\bar{w} \to \bar{w} - F(L^{-1}\bar{a}) \to L[\bar{w} - F(L^{-1}\bar{a})]. \end{aligned}$$

Lemma 4.3. *Let $F \in \mathfrak{so}(1,3)$, and let $\tilde{F}_{cd} = \frac{1}{2}\varepsilon_{abcd}F^{ab}$, then*

$$\tilde{F}_{cd}F^{ce} = \frac{1}{4}(F^{ab}\tilde{F}_{ab})\delta^e_d$$

Proof.

$$\begin{aligned} \tilde{F}_{\gamma\delta}F^{\gamma\sigma} &= \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\sigma} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}[-\frac{1}{2}\varepsilon^{\gamma\sigma\eta\nu}\tilde{F}_{\eta\nu}] = -\frac{1}{4}F^{\alpha\beta}\tilde{F}_{\eta\nu}(-\delta^{\sigma\eta\nu}_{\alpha\beta\delta}) \\ &= \frac{1}{4}F^{\alpha\beta}\tilde{F}_{\eta\nu}(\delta^{\eta\nu}_{\alpha\beta}\delta^{\sigma}_{\delta} + \delta^{\sigma\eta}_{\alpha\beta}\delta^{\nu}_{\delta} + \delta^{\nu\sigma}_{\alpha\beta}\delta^{\eta}_{\delta}) = \frac{1}{2}(F^{\eta\nu}\tilde{F}_{\eta\nu}\delta^{\sigma}_{\delta} + F^{\sigma\eta}\tilde{F}_{\eta\delta} + F^{\nu\sigma}\tilde{F}_{\delta\nu}) \\ &= -\tilde{F}_{\gamma\delta}F^{\gamma\sigma} + \frac{1}{2}(F^{\alpha\beta}\tilde{F}_{\alpha\beta})\delta^{\sigma}_{\delta} \end{aligned}$$

□

Theorem 4.4. *Let $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\mathsf{T}} & 0 \end{pmatrix}$ be an element of $\mathfrak{iso}(1,3)$ and let us suppose that $I_2 := -\frac{1}{2}\tilde{F}_{cd}F^{cd} = 0$, then*

$$I_3 := \tilde{F}_{ab}w^b\tilde{F}^a{}_c w^c$$

is Ad-invariant.

Condition $F \neq 0$ is Ad-invariant. If $I_2 = 0$ the condition $\tilde{F}\bar{w} \neq 0$ is Ad-invariant. If $I_2 = 0$, $I_1 > 0$, $I_3 \geq 0$, $\tilde{F}\bar{w} \neq 0$, then $\epsilon_1 = sgn(\tilde{F}^0{}_a\tilde{F}^a{}_b w^b)$ is a well defined Ad-invariant. If $I_2 = 0$, $I_1 > 0$, $I_3 \leq 0$, $\tilde{F}\bar{w} \neq 0$, then $\epsilon_2 = sgn(\tilde{F}^0{}_c w^c)$ is a well defined Ad-invariant.

If $I_2 = I_1 = 0$ but $F \neq 0$, then I_3 is non-negative. The equality $I_3 = 0$ holds if and only if $\bar{w} \in KerF^2$ (this statement is Ad-invariant). Let us consider the cases $I_3 > 0$ and $I_3 = 0$.

If $I_3 > 0$ then $\epsilon_1 = sgn(\tilde{F}^0{}_a \tilde{F}^a{}_b w^b)$ is well defined and provides another Ad-invariant.

Let us come to $I_3 = 0$. Since $F^3 = 0$, we have $ImF \subset KerF^2$. Frame translations send $\bar{w}$ to another element in the same class of $KerF^2/ImF$. There is a choice that minimizes ${w'}^a w'_a$. This (non-negative) minimum

$$I_4 := \min_{\bar{w}' \in [\bar{w}]} {w'}^a {w'}_a,$$

is a characteristic of the class and is, therefore, an Ad-invariant. (Any minimizing element belongs to $KerF$.)

Finally, if $F = 0$ (and hence $I_2 = I_1 = 0$), then

$$I_4 := w^a w_a,$$

is Ad-invariant. Furthermore, if $I_4 < 0$, then $\epsilon_1 = sgn\, w^0$ is Ad-invariant.

Remark 4.5. The equation defining I_4 for $F = 0$ follows from an extension of that defining I_4 for $I_1 = I_2 = I_3 = 0$, $F \neq 0$. Indeed, if $F = 0$, $\mathrm{Ker}F^2 = W$ and $\mathrm{Im}F = \{0\}$ thus every class contains only one element, i.e. $\mathrm{Ker}F^2/\mathrm{Im}F = W$, thus $\min_{\bar{w}' \in [\bar{w}]} {w'}^a {w'}_a = w^a w_a$.

Proof. It is sufficient to prove that I_2 is invariant under (i) homogeneous transformations, i.e., $\bar{a} = \bar{0}$, and (ii) translations, i.e., $L = I$. Case (i) is clear since I_3 is defined as the (Lorentzian) square of a 4-vector and, under the assumption $\bar{a} = \bar{0}$, both $F^a_{\ b}$ and w^b transform as tensors. For case (ii) observe that $F \to F$ and $w^b \to w^b - F^b{}_c a^c$, and by lemma 4.3, I_3 is left invariant.

Condition $F \neq 0$ is trivially Ad-invariant. If $I_2 = 0$ condition $\tilde{F}\bar{w} \neq 0$ is invariant under frame changes because of lemma 4.3. Suppose that $I_2 = 0$, $I_1 > 0$, $I_3 \geq 0$, and $\tilde{F}\bar{w} \neq 0$ if $I_3 = 0$. Let $z^c := \tilde{F}^c{}_a \tilde{F}^a{}_b w^b$. There is a frame for which $z^0 \neq 0$, and z^γ is null, namely that for which $F = A$ with $\varphi = 0$. Under translations of the frame z^c does not change, while under homogeneous transformations it behaves as a vector, thus it is a lightlike vector in any frame and $\epsilon_1 := \mathrm{sgn}(z^0)$ is invariant.

Suppose that $I_2 = 0$, $I_1 > 0$, $I_3 \leq 0$, and $\tilde{F}\bar{w} \neq 0$ if $I_3 = 0$. Let $v^a := \tilde{F}^a{}_c w^c$. Since $I_3 \leq 0$, there is a frame for which $v^0 \neq 0$, namely that for which $F = A$ with $\varphi = 0$. In that frame, since $I_3 \leq 0$, v^c is a causal vector. Because of lemma 4.3, under translations of the frame v^c is left unaltered, while under homogeneous transformations it transform as a tensor, thus in any frame v^c is a causal vector and $\epsilon_2 := \mathrm{sgn}\, v^0$ is invariant.

Suppose $I_2 = I_1 = 0$ and $F \neq 0$. The inequality $I_3 \geq 0$ can be easily checked for $F = B$ and arbitrary $\bar{w}$. Since I_3 is a scalar under transformations of type (i), the inequality holds for any F in the orbit of B. With the same type of argument, i.e. studying the case $F = B$, we can show that $\tilde{F}^2 = F^2$, and from the special form of B^2 we easily deduce that $I_3 = 0$ if and only if $\bar{w} \in \mathrm{Ker}F^2$.

Let us observe that the condition $\bar{w} \in \mathrm{Ker}F^2$ is independent of the frame since under translations of the frame $\bar{w}$ is added terms belonging to $\mathrm{Im}F$, and $\mathrm{Im}F \subset \mathrm{Ker}F^2$ as $F^3 = 0$.

Let us consider $\epsilon_1 := \operatorname{sign} v^0$ where $v^c := \tilde{F}^c{}_a \tilde{F}^a{}_b w^b = F^c{}_a F^a{}_b w^b$. Let us observe that v^c is a null vector because $F^3 = 0$. The frame for which $F = B$ shows that if $\bar{w} \notin \mathrm{Ker} F^2$ (iff $I_3 > 0$) then $v^0 \neq 0$ in that frame. Under homogeneous transformations of the frame v^c transform as a tensor (hence remaining a null vector), while under translations of the frame it is left unchanged because $F^3 = 0$. As a consequence, ϵ_1 is well defined and invariant.

Let us consider the possibility $I_3 = 0$, and hence $\bar{w} \in \mathrm{Ker} F^2$. We mentioned that under frame changes $\bar{w}$ transforms as $\bar{w} \to \bar{w} - F\bar{a}$, thus being altered by additive terms belonging to $\mathrm{Im} F$. Let us show that this additive term can be chosen so as to minimize $w^a w_a$. Let us study the problem in the frame for which $F = B$ so that: $\mathrm{Ker} F = \mathrm{Span}(e_0 + e_1, e_3)$, $\mathrm{Ker} F^2 = \mathrm{Span}(e_0 + e_1, e_2, e_3)$, $\mathrm{Im} F = \mathrm{Span}(e_0 + e_1, e_2)$, $\mathrm{Im} F^2 = \mathrm{Span}(e_0 + e_1)$. Then $w = (l, l, c, d)$, $w^a w_a = c^2 + d^2$, where l and c can be chosen freely. Clearly the minimum exists, and is attained for $c = 0$ (l remains undetermined). Any minimizing vector, being of the form $w = (l, l, 0, d)$, belongs to $\mathrm{Ker} F$.

The last statement is trivial.

□

Definition 4.6. We call ϵ_1 the *time orientation* of the Lie algebra element (future directed or positive if $\epsilon_1 = +1$). We call ϵ_2 *helicity*.

Theorem 4.7. *(Classification of Lie orbits)*
Let $\mathcal{P} \in \mathfrak{IL}$, let $I_1, I_2, I_3, I_4, \epsilon_1, \epsilon_2$, be the Ad-invariants of $\mathcal{P}$ and let φ and θ be defined as in Eqs. (6)-(7). Moreover, if $I_2 = 0$ and $I_1 \neq 0$ let

$$b = \sqrt{\left|\frac{I_3}{2I_1}\right|}.$$

Then it is possible to choose the reference frame in such a way that $\mathcal{P}$ takes one of the following matrix forms, and corresponding vector field form.

1. $I_2 \neq 0$:

$$\begin{pmatrix} 0 & -\varphi & 0 & 0 & 0 \\ -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \varphi(x^0 \partial_1 + x^1 \partial_0) + \theta(x^2 \partial_3 - x^3 \partial_2),$$

2. $I_2 = 0$, $I_1 < 0$ ($I_3 \geq 0$):

$$\begin{pmatrix} 0 & -\varphi & 0 & 0 & 0 \\ -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \varphi(x^0 \partial_1 + x^1 \partial_0) + b\partial_3,$$

3. $I_2 = 0$, $I_1 > 0$, $I_3 < 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_2 b \\ 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta(x^2 \partial_3 - x^3 \partial_2) + \epsilon_2 b \partial_1,$$

4. $I_2 = 0,\ I_1 > 0,\ I_3 > 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1 b \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta(x^2\partial_3 - x^3\partial_2) + \epsilon_1 b\partial_0,$$

5. $I_2 = 0,\ I_1 > 0,\ I_3 = 0,\ \tilde{F}\bar{w} \neq 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta(x^2\partial_3 - x^3\partial_2) + \epsilon_1\partial_0 + \epsilon_2\partial_1,$$

6. $I_2 = 0,\ I_1 > 0,\ I_3 = 0,\ \tilde{F}\bar{w} = 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta(x^2\partial_3 - x^3\partial_2),$$

7. $I_2 = I_1 = 0,\ F \neq 0,\ I_3 > 0$:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & -\epsilon_1\sqrt{I_3} \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (x^0\partial_2 + x^2\partial_0) + (x^2\partial_1 - x^1\partial_2) + \epsilon_3\sqrt{I_3}\,\partial_0,$$

8. $I_2 = I_1 = 0,\ F \neq 0,\ I_3 = 0$:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{I_4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (x^0\partial_2 + x^2\partial_0) + (x^2\partial_1 - x^1\partial_2) + \sqrt{I_4}\,\partial_3,$$

9. $F = 0,\ I_4 < 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1\sqrt{-I_4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_1\sqrt{-I_4}\,\partial_0,$$

10. $F = 0,\ I_4 > 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{I_4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sqrt{I_4}\,\partial_3,$$

11. $F=0$, $I_4=0$, $\mathcal{P}\neq 0$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_1(\partial_0+\partial_1).$$

Finally, there is a twelfth case corresponding to the trivial Lie algebra orbit of the zero element of $\mathfrak{IL}$.

Stated in another way, the orbits of the adjoint action of $ISO(1,3)^{\uparrow}$ on $\mathfrak{iso}(1,3)$ admit one and only one of the twelve representatives given above.

Proof. If $F=0$ then w^α transform as a vector under changes of frame. Cases 9, 10 and 11 are then rather obvious, it is sufficient to observe that under a rotation of the frame we can accomplish $w^2=w^3=0$, thus though a boost on the timelike plane $Span(e_0,e_1)$ we can obtain one of the forms 9, 10 or 11 (or the trivial zero element).

Thus let $F\neq 0$. If $I_2\neq 0$ then F is non-singular thus we find a suitable translation $\bar{w}\to\bar{w}-F\bar{a}$ which sends $\bar{w}$ to zero. Then with a homogeneous transformation we send F to the canonical form A of theorem 3.1. We obtain in this way the representative of case 1.

Suppose that $F\neq 0$, $I_2=0$, $I_1\neq 0$. According to theorem 3.1 through a homogeneous transformation of the reference frame we can send F to the canonical form A. If $I_1<0$ then $\theta=0$, if $I_1>0$ then $\varphi=0$.

In the former case $A|_{\mathrm{Span}(e_0,e_1)}:\mathrm{Span}(e_0,e_1)\to\mathrm{Span}(e_0,e_1)$ is non-singular thus with a translation of the reference frame $\bar{w}\to\bar{w}-A\bar{a}$ we accomplish $w^0=w^1=0$. With a rotation of the reference plane on the plane $\mathrm{Span}(e_2,e_3)$ we obtain $w^2=0$. Finally, with a rotation of π along the first axis we choose suitably the sign of w^3 so as to send it to $-b$ (by the existence of the invariants). We arrive in this way at the representative 2.

Let us consider the latter case $I_1>0$, $\varphi=0$. The map $A|_{\mathrm{Span}(e_2,e_3)}$: $\mathrm{Span}(e_2,e_3)\to\mathrm{Span}(e_2,e_3)$ is non-singular, thus with a translation of the reference frame $\bar{w}\to\bar{w}-A\bar{a}$ we accomplish $w^2=w^3=0$. Then with a boost in the timelike plane $\mathrm{Span}(e_0,e_1)$ we accomplish one of the representatives 3,4, 5 or 6.

Suppose that $I_2=I_1=0$, $\bar{w}\notin \mathrm{Ker}F^2$. With a homogeneous transformation of the reference frame we send F to B for some $\alpha>0$. The image of B is $\mathrm{Span}(e_0+e_1,e_2)$ thus with a translation of the reference frame we obtain $w^1=w^2=0$. Since $F=B$ the condition $\bar{w}\notin\mathrm{Ker}F^2$ implies $w^0\neq 0$.

With a boost in the timelike plane $\mathrm{Span}(e_0,e_1)$ of rapidity r followed by a translation we send α to $\alpha':=\alpha e^{-r}$ and w^0 to w^0e^r, keeping $w^1=w^2=0$.

Thus we can choose r so that $|w^0 e^r| > |w^3|$. Next we use the identity

$$\begin{pmatrix} \cosh\gamma & 0 & 0 & -\sinh\gamma & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\sinh\gamma & 0 & 0 & \cosh\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\alpha' & 0 & -w^0 e^r \\ 0 & 0 & -\alpha' & 0 & 0 \\ -\alpha' & \alpha' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -w^3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh\gamma & 0 & 0 & \sinh\gamma & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sinh\gamma & 0 & 0 & \cosh\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & -\alpha'\cosh\gamma & 0 & -(w^0 e^r \cosh\gamma - w^3 \sinh\gamma) \\ 0 & 0 & -\alpha' & 0 & 0 \\ -\alpha'\cosh\gamma & \alpha' & 0 & -\alpha'\sinh\gamma & 0 \\ 0 & 0 & \alpha'\sinh\gamma & 0 & (w^0 e^r \sinh\gamma - w^3\cosh\gamma) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which followed by a rotation of angle $\beta = \tan^{-1}\sinh\gamma$ around e_2 brings the matrix to the following form (note that $\sin\beta = \tanh\gamma$, $\cos\beta = 1/\cosh\gamma$)

$$\begin{pmatrix} 0 & 0 & -\alpha'\cosh\gamma & 0 & -(w^0 e^r \cosh\gamma - w^3\sinh\gamma) \\ 0 & 0 & -\alpha'\cosh\gamma & 0 & -\sinh\gamma(w^0 e^r \tanh\gamma - w^3) \\ -\alpha'\cosh\gamma & \alpha'\cosh\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w^0 e^r \tanh\gamma - w^3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choosing $\gamma = \tanh^{-1}(\frac{w^3}{w^0 e^r})$ we arrive at

$$\begin{pmatrix} 0 & 0 & -\alpha'\cosh\gamma & 0 & -w^0 e^r/\cosh\gamma \\ 0 & 0 & -\alpha'\cosh\gamma & 0 & 0 \\ -\alpha'\cosh\gamma & \alpha'\cosh\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, with a boost on the timelike plane $\mathrm{Span}(e_0, e_1)$ followed by a translation we obtain

$$\begin{pmatrix} 0 & 0 & -\alpha & 0 & -w^0 \\ 0 & 0 & -\alpha & 0 & 0 \\ -\alpha & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus with a sequence of frame changes we have been able to send w^3 to zero. With a final boost on the timelike plane $\mathrm{Span}(e_0, e_1)$ followed by a translation we send α to ± 1 and w^0 to $\epsilon_3\sqrt{I_3}$ keeping unchanged all the other matrix entries. A last π-rotation on the plane $\mathrm{Span}(e_2, e_3)$ sends the possible value $\alpha = -1$ to $\alpha = 1$ giving us the representative 7.

Suppose that $I_2 = I_1 = 0$, $\bar{w} \in \mathrm{Ker} F^2$. With a homogeneous transformation of the reference frame we send F to B with $\alpha = 1$. At the end of the proof of theorem 4.4 we have shown that the reference frame can be translated in such a way that $\bar{w} = (l, l, 0, d)$ where $d^2 = I_4$. With another translation we obtain $l = 0$ and if $d \leq 0$, with a π-rotation on the plane $\mathrm{Span}(e_2, e_3)$ we obtain $d \geq 0$ and hence $d = \sqrt{I_4}$, which gives us the representative 8.

□

Corollary 4.8. *Let k be a Killing field on Minkowski spacetime. Then there is a reference frame through whose coordinates k takes one of the twelve forms listed in theorem 4.7.*

Proof. This is just a rephrasing of the previous theorem, given that $\mathfrak{IL}$ is the Lie algebra of the Killing fields of Minkowski spacetime. $\square$

Remark 4.9. The closure of the conjugacy class 8 of theorem 4.7 contains class 10. Indeed, a boost of the frame in the timelike plane $\mathrm{Span}(e_0, e_1)$ shows that

$$\begin{pmatrix} 0 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 \\ -\alpha & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{I_4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for any $\alpha > 0$ stays in class 8. Taking the limit $\alpha \to 0$ we obtain the representative of class 10. As a consequence, no continuous *Ad*-invariant function $f : \mathfrak{IL} \to \mathbb{R}$ can distinguish between classes 8 and 10. In particular, no algebraic Ad-invariant built from the pair $(F^a_{\ b}, w^b)$ can allow us to distinguish between the two classes.

Remark 4.10. In a recent paper Barbot considered the conjugacy classes of the proper orthochronous inhomogeneous Lorentz group [5, Sect. 6] and obtained a, somewhat coarser, classification. With respect to that work our proofs are slightly longer because our aim was to obtain nice representatives by bringing the homogeneous and translational part into a canonical form. Thanks to our complete set of Ad-invariants we are able to identify a single conjugacy class for each choice of allowed Ad-invariants, and we are able to tell exactly which is the conjugacy class of a given transformation by means of straightforward matrix calculations. On the other hand, the more geometrical approach by Barbot serves more easily the intuition for the sake of the classification.

Barbot selects some families of conjugacy classes which, although we worked on the Lie algebra and he on the Lie group, can be put into correspondence with our families. The correspondence is as follows.

Elliptic**:** These are our cases 3-6, which correspond to $I_2 = 0$, $I_1 > 0$, and the pure translations 9-11.

Hyperbolic**:** This is our case 2, which corresponds to $I_2 = 0$, $I_1 < 0$.

Unipotent**:** These are our cases 7-8, which correspond to $I_2 = I_1 = 0$, $F \neq 0$, with $I_3 > 0$ for 7 and $I_3 = 0$ for 8. Barbot's trichotomy is as follows. The *linear* case is our case 8 with $I_4 = 0$. The *tangent* case is our case 8 with $I_4 \neq 0$. The *transverse* case is our case 7.

Loxodromic**:** This is our case 1 which corresponds to $I_2 \neq 0$.

Parabolic**:** Does not apply in the four dimensional spacetime case considered here.

4.5 The Lie wedge

In section 4.1 we argued that the semigroup $I \subset ISO(1,3)^{\uparrow}$ (resp. J) selects those transformations that are physically reasonable, in the sense that they can be induced by the dragging of an observer's frame on spacetime.

We would like to select those generators that induce the mentioned transformation belonging to I (resp. J). In other words, we have to find the Lie algebra counterpart of the semigroup. Fortunately, there is a well developed Lie theory for subsemigroups of Lie groups [19, 20]. If S is a closed subsemigroup of a Lie group G, its Lie wedge (or cone) is the set

$$L(S) = \{X \in \mathfrak{g} : \exp(\mathbb{R}^+ X) \subset S\}, \tag{21}$$

where $\mathbb{R}^+ = (0, +\infty)$. The Lie cone is convex because of the following identity which can be deduced from the Baker-Campbell-Hausdorff formula [19, Lemma II.1.1]

$$\exp[X+Y] = \lim_{n \to +\infty} [\exp \frac{X}{n} \exp \frac{Y}{n}]^n.$$

The semigroup J is closed, thus the standard theory which can be found in [19, 20] applies to it. In particular, the *causal wedge* $L(J)$ is convex.

Remark 4.11. Although I is not closed, we shall define $L(I)$ according to Eq. (21) and we shall call it the *timelike wedge*. The reader is warned that we are making an abuse of notation, and that $L(I)$ is not convex.[1]

Let us identify the Lie wedges for the semigroups I and J.

Theorem 4.12. *The Lie wedges of the semigroups I and J satisfy*

$$L(J) = \{\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} : \ F \in \mathfrak{so}(1,3) \ \textit{and } \bar{w} \textit{ is f.d. nonspacelike}\}, \tag{22}$$

$$L(J)\backslash L(I) = \{\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} : \ F \in \mathfrak{so}(1,3), \ \bar{w} \textit{ is f.d. null and } F\bar{w} = \lambda \bar{w}\}, \tag{23}$$

$$e^{[L(J)\backslash L(I)]} = \{\begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} : \ \Lambda \in SO(1,3), \ \bar{b} \textit{ is f.d. null and } \Lambda \bar{b} = e^{\lambda}\, \bar{b}\}, \tag{24}$$

$$\subsetneq J \backslash I, \tag{25}$$

$$e^{L(J)} \subsetneq J, \tag{26}$$

where $\lambda \in \mathbb{R}$.

Proof. Let us consider the system which defines the exponential map (16)-(17) with initial condition $\Lambda = I$, $\bar{b} = \bar{0}$. If $\begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}(s)$ belongs to J for all $s > 0$ then the same holds for small positive s. By Eq. (17), since at $s = 0$, $\bar{b} = \bar{0}$, we have that $\bar{w}$ must be f.d. nonspacelike.

[1]There is a definition of Lie wedge that applies to non-closed semigroup [19, 20], but it would lead back to $L(J)$, while we will need $L(I)$ for our arguments.

Conversely, let us suppose that $\bar{w}$ is f.d. nonspacelike, and let $\bar{c}$ be such that $\bar{b} = \Lambda\bar{c}$ (thus $\bar{b}$ is f.d. nonspacelike iff $\bar{c}$ is). Eq. (17) becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\bar{c} = \Lambda^{-1}\bar{w}. \tag{27}$$

Since the right-hand side is nonspacelike, the integral $\bar{c}$ is nonspacelike. Equation (22) is proved.

Let us prove Eq. (23). Let us suppose that $\bar{b}(s)$ is f.d. nonspacelike for all $s > 0$ and lightlike for some $\tilde{s} > 0$. The same holds for $\bar{c}(s)$. We already know that $\bar{w}$ must be nonspacelike and equation (27) proves that $\bar{c}(s)$ is a smooth causal curve or $\bar{c}(s) = \bar{w} = \bar{0}$ for all s. Every causal curve which is not a lightlike pregeodesic connects chronologically related points [18]. Thus $\bar{c}(s)$, $0 \le s < \tilde{s}$ is a null pregeodesic curve or $\bar{c}(s) = \bar{0}$. Imposing that the tangent vector to $\bar{c}(s)$ be proportional to the same null vector for all $0 \le s < \tilde{s}$ gives $\Lambda^{-1}(s)\bar{w} = f(s)\bar{n}$, for some smooth function $f(s)$. This equation for $s = 0$ gives $\bar{w} = f(0)\bar{n}$ which shows that $\bar{w}$ is null. Let us differentiate $\bar{w} = f(s)\Lambda(s)\bar{n}$ and evaluate it at $s = 0$. We get $0 = f'(0)\bar{n} + Ff(0)\bar{n}$ which proves that w is an eigenvector of F. Let λ be the eigenvalue, i.e. $F\bar{w} = \lambda\bar{w}$. The scalar product of Eq. (17) with $\bar{w}$ gives, $\frac{\mathrm{d}}{\mathrm{d}s}\eta(w,b) = -\lambda\eta(w,b)$, and using the initial condition $\bar{b}(0) = 0$ we obtain $\eta(w,b) = 0$. Since, by assumption, $\bar{b}$ is f.d. nonspacelike, we have $\bar{b} = h(s)\bar{w}$ which plugged back into Eq. (17) gives $h' = \lambda h + 1$ or $\bar{w} = \bar{0}$. If $\bar{w} \neq \bar{0}$ we infer $h(s) = \frac{1}{\lambda}[\exp(\lambda s) - 1]$ for $\lambda \neq 0$ and $h(s) = s$ for $\lambda = 0$. In summary, if the matrix $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ belongs to $L(J)\backslash L(I)$ then $\begin{pmatrix} \Lambda(s) & -\bar{b}(s) \\ \bar{0}^{\intercal} & 1 \end{pmatrix} = \exp[s\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}]$ is such that $\bar{b}(s) = \frac{1}{\lambda}[\exp(\lambda s) - 1]\hat{w}$ where it is understood that $\frac{1}{\lambda}[\exp(\lambda s) - 1] := s$ for $\lambda = 0$. In particular, $\bar{b}(s)$ is f.d. null for all s and it is an eigenvector for Λ with positive eigenvalue because

$$\Lambda\bar{b} = (\exp F)\,\bar{b} = (\exp\lambda)\bar{b}.$$

(Notice that with such a $\bar{b}$ the matrix $\begin{pmatrix} \Lambda(s) & -\bar{b}(s) \\ \bar{0}^{\intercal} & 1 \end{pmatrix}$ belongs to $J\backslash I$.) Let us show that conversely every matrix $\begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}$ belongs to $\exp[L(J)\backslash L(I)]$ provided $\bar{b}$ is f.d. null and it is an eigenvector with positive eigenvalue for Λ. The Lie group $SO(1,3)^{\uparrow}$ is exponential, namely the exponential map is surjective (for the references see after theorem 3.6). Thus there is some $F \in \mathfrak{so}(1,3)$ such that $\Lambda = \exp F$. Furthermore, Prop. 3.5 shows that Λ and F have the same f.d. null eigenvectors thus $F\bar{b} = \lambda\bar{b}$. Let us define $\bar{w} = \lambda(\exp\lambda - 1)^{-1}\bar{b}$ for $\lambda \neq 0$, and $\bar{w} = \bar{b}$ for $\lambda = 0$, then by the above calculations $\exp\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} = \begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}$. We proved Eq. (24). The fact that the inclusion (25) is strict follows taking $\begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}$ such that $\Lambda \neq I$ (thus some null vectors are not eigenvectors) and $\bar{b}$ is a f.d. null vector which is not an eigenvector. This matrix belongs to $J\backslash I$ but not to $\exp[L(J)\backslash L(I)]$. The last strict inclusion is an immediate consequence of the previous one and $\exp I \subset I$. $\square$

As a simple corollary of the previous theorem we obtain

Proposition 4.13. *The sets $L(J)$ and $L(J)\backslash L(I)$ are closed and $L(I)$ is not open. However, $L(I)$ is open in the topology induced on $L(J)$.*

4.5.1 The strict inclusion $\exp L(I) \subsetneq I$ and the causal cone of F

Suppose that $F \in \mathfrak{so}(1,3)$ is so close to zero that defined $\Lambda = \exp F \in SO(1,3)^{\uparrow}$ there is no other $F' \in \mathfrak{iso}(1,3)$ such that $\Lambda = \exp F'$. We ask the following question: for which $\bar{b} \in \mathbb{R}^4$ we have $\begin{pmatrix} \Lambda & -\bar{b} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \in \exp L(I)$? Is it possible to find some $\bar{b}$ such that this matrix belongs to I but not to $\exp L(J)$? According to theorem 4.2 the $\bar{b}$s which satisfy the first condition are those which are causal according to the metric

$$G = (\frac{F}{e^F - I})^{\intercal} \eta \, (\frac{F}{e^F - I}),$$

and f.d. timelike according to η (recall that $\exp L(I) \subset I$, then $\bar{w} = \frac{F}{e^F - I}\bar{b}$ cannot be p.d. timelike for otherwise $\bar{b}$ would be p.d. timelike because of Eq. (18)).

It is instructive to calculate this metric for the canonical forms A and B of F given by theorem 3.1. The result is

$$G(A) = \begin{pmatrix} \frac{-\varphi^2}{2\cosh\varphi - 2} & 0 & 0 & 0 \\ 0 & \frac{\varphi^2}{2\cosh\varphi - 2} & 0 & 0 \\ 0 & 0 & \frac{\theta^2}{2 - 2\cos\theta} & 0 \\ 0 & 0 & 0 & \frac{\theta^2}{2 - 2\cos\theta} \end{pmatrix}$$

$$G(B) = \begin{pmatrix} -1 + \frac{\alpha^2}{12} & -\frac{\alpha^2}{12} & 0 & 0 \\ -\frac{\alpha^2}{12} & 1 + \frac{\alpha^2}{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the generic F we have $G(F) = LG(A)L^{-1}$ or $G(F) = LG(B)L^{-1}$, where $L \in SO(1,3)^{\uparrow}$ and the former or the latter case apply depending on whether F belongs to the Ad-orbit of A or B. Using the inequalities $\frac{\varphi^2}{2\cosh\varphi - 2} \le 1$ (with equality iff $\varphi = 0$) and $\frac{\theta^2}{2 - 2\cos\theta} \ge 1$ (with equality iff $\theta = 0$), we easily infer that if F is in the Ad-orbit of A, then the causal cone of $G(F)$ is contained inside the causal cone of η. Moreover, if $F \ne 0$ it is tangent to it in just two distinct null directions. As a consequence, the set $I \backslash \exp L(J)$ is non-empty, it suffices to consider a vector $\bar{b}$ which stay outside the causal cone of $G(F)$ but inside the timelike cone of η. Actually, we can say more, namely that $\overline{\exp J} \subsetneq J$, because under small perturbations of F and of $\bar{b}$ as above, $\bar{b}$ keeps staying outside the causal cone of $G(F)$.

In order to complete our analysis, observe that if $y, x \in \mathbb{R}$ are such that $y^2 \ge x^2$ then

$$(-1 + \frac{\alpha^2}{12})x^2 - \frac{\alpha^2}{6}xy + (1 + \frac{\alpha^2}{12})y^2 \ge \frac{\alpha^2}{12}(x - y)^2,$$

which implies that whenever F belongs to the Ad-orbit of B, the causal cone of $G(F)$ is contained in the causal cone of η and it is tangent to it in just one null direction. As a consequence, we can again conclude that the set $I \backslash \exp L(J)$ is non-empty.

We summarize some of these findings through the following proposition.

Proposition 4.14. *We have* $\exp L(I) \subsetneq I$ *and* $I \backslash \exp L(J) \neq \emptyset$. *Moreover,* $\overline{\exp J} \subsetneq J$, *that is,* J *is not weakly exponential [19, 20, 21].*

Remark 4.15. One of the consequences of the strict inclusion $\exp L(I) \subsetneq I$ is that, given two events $p, q \in M$, with $q \in J^{+}(p)$, and two proper orthochronous bases $\{e_a^p\}$, $\{e_a^q\}$, at p and q respectively, it is possible that no observer which moves with constant acceleration and angular velocity can start with a comoving base coincident with $\{e_a^p\}$ to later reach $\{e_a^q\}$. One of the points of this paper is to show that, nevertheless, $\{e_a^p\}$ can be dragged into $\{e_a^q\}$, with the motion of an observer which moves with constant acceleration and angular velocity. However, this observer does not necessarily pass through p or q.

In the next section we study the physical meaning of these causal orbits on the Lie algebra.

4.6 The causal orbits

The Ad action of $ISO(1,3)^{\uparrow}$ on $\mathfrak{iso}(1,3)$ (or the Ad action of $IL_{+}^{\uparrow}$ on $\mathfrak{IL}$) generates orbits which we classified in section 4.4.

It is possible to assign a causal character to these orbits.

Theorem 4.16. *The Lie algebra Ad-orbits on* $\mathfrak{iso}(1,3)$ *which admit some representative in* $L(I)$ *belong to the families of orbits (according to the classification of theorem 4.7) 1, 2, 4, 7, 9, with* $\epsilon_1 = 1$ *(whenever it applies). Those which admit some representative in* $L(J) \backslash L(I)$ *but do not admit any representative in* $L(I)$ *belong to the families of orbits 5, 6, 11, with* $\epsilon_1 = 1$, *8 with* $I_4 = 0$, *and the trivial orbit of the origin (12).*

Proof. If we start from representatives 1 or 2 in theorem 4.7, then, since $\mathrm{Im} F = \mathrm{Span}(e_0, e_1)$, with a translation of the frame we can send $w^0 = 0$ to $w^0 = c$, with $c > 0$ arbitrary (in particular $c > b$ in case 2), and leaving unaltered all the other matrix entries. After this translation w^a becomes f.d. timelike thus the new representative belongs to $L(I)$. Representatives 4,7,9, with $\epsilon_1 = 1$ satisfy w^a f.d. timelike, thus there is nothing to prove. As for representatives 5,11, with $\epsilon_1 = 1$, 6, or 8 with $I_4 = 0$, it is clear that $\bar{w}$ is f.d. null and that $\bar{w}$ is an eigenvector of F.

It remains to show that orbits of type 3, 8 with $I_4 \neq 0$, 10, do not have any representative in $L(J)$, that those of type 4, 5, 7, 9, 11, with $\epsilon_1 = -1$, do not have any representative in $L(J)$, and that those of type 5, 11, with $\epsilon_1 = 1$, 6, 8 with $I_4 = 0$, 12, do not have any representative in $L(I)$.

The argument is the same for most of these cases. Any frame change can be accomplished with a translation followed by a homogeneous transformation.

In cases 3, 8 with $I_4 \neq 0$, 10, and 4, 5, 9, 11, with $\epsilon_1 = -1$, $\bar{w}$ is not f.d. non-spacelike and ImF is a spacelike subspace orthogonal to it (possibly empty). After the first translation of the frame, the new $\bar{w}$ becomes the sum of the old $\bar{w}$ and of an element belonging to ImF and hence, is still non f.d. non-spacelike.

As for case 8 with $I_4 \neq 0$, any frame change can be accomplished with a translation followed by a homogeneous transformation. The former transformation does not change the spacelike causal character of $\bar{w}$ (since one gets $w^0 = w^1$ and possibly $w^3 \neq 0$ for any choice of $\bar{a}$) while the latter preserves its causal character.

Analogously, in case 7 with $\epsilon_1 = -1$, it is easy to check that operating with a translation to make w^0 positive leads to w^a spacelike.

The proof that classes 5, 11, with $\epsilon_1 = 1$, 6, 8 with $I_4 = 0$ and 12, do not have any representative in $L(I)$, proceeds analogously.

□

Theorem 4.17. *Let $\mathcal{P} \in \mathfrak{IL}$ and suppose that for some $q \in M$, $(\exp \mathcal{P})q \in J^+(q)$ (resp. $(\exp \mathcal{P})q \in I^+(q)$), then there is some $q' \in M$ such that $\exp(\mathcal{P}s)q' \in J^+(q')$ (resp. $\exp(\mathcal{P}s)q' \in I^+(q')$) for every $s > 0$.*

Stated in another way, if an element of $\mathfrak{iso}(1,3)$ has exponential belonging to J (resp. I), then there must be some representative in its Ad-orbit which belongs to $L(J)$ (resp. $L(I)$).

Proof. Suppose that $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} \in \mathfrak{iso}(1,3)$ has exponential belonging to J (resp. I). There is a matrix $\begin{pmatrix} L & -\bar{a} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \in ISO(1,3)^{\uparrow}$ such that

$$\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} = \begin{pmatrix} L & -\bar{a} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} \begin{pmatrix} L & -\bar{a} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} L & \bar{0} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} I & -L^{-1}\bar{a} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} \begin{pmatrix} I & -L^{-1}\bar{a} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}^{-1} \begin{pmatrix} L & \bar{0} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}^{-1},$$

where $\begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ is one of the representatives of theorem 4.7. Let $\bar{c} = L^{-1}\bar{a}$

$$\exp \begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} = \begin{pmatrix} L & \bar{0} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} I & -\bar{c} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} (\exp \begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}) \begin{pmatrix} I & -\bar{c} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}^{-1} \begin{pmatrix} L & \bar{0} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}^{-1}.$$

The frame changes obtained through homogeneous transformations send J (resp. I) to itself, thus the assumption of the theorem is that

$$\begin{pmatrix} I & -\bar{c} \\ \bar{0}^{\intercal} & 1 \end{pmatrix} (\exp \begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}) \begin{pmatrix} I & -\bar{c} \\ \bar{0}^{\intercal} & 1 \end{pmatrix}^{-1},$$

belongs to J (resp. I). Let $\begin{pmatrix} \check{\Lambda}(s) & -\check{b}(s) \\ \bar{0}^{\intercal} & 1 \end{pmatrix} = \exp(\begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} s)$ then we are assuming that

$$\bar{r}(s) := -(\check{\Lambda}(s) - I)\bar{c} + \check{b}(s),$$

is f.d. nonspacelike (resp. timelike) for some $\bar{c}$ and for $s = 1$. Let us use Eqs. (16)-(17)

$$\frac{\mathrm{d}}{\mathrm{d}s}(\bar{r}(s) - \check{w}s) = \check{F}(\bar{r}(s) - \bar{c}), \qquad \text{and } \bar{r}(0) = \bar{0},$$

from which we obtain $\bar{r}(1) \in \check{w} + \mathrm{Im}\check{F}$. This inclusion implies that $\begin{pmatrix} \check{F} & -\check{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ belongs to the same orbit of $\begin{pmatrix} \check{F} & -\bar{r}(1) \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ (they are connected through a translation of the frame), which, because of the causal character of $\bar{r}(1)$, belongs to $L(J)$ (resp. $L(I)$). Thus the Ad-orbit of $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ contains an element in $L(J)$ (resp. $L(I)$).

□

Definition 4.18. A conjugacy class of $ISO(1,3)^{\uparrow}$ is *causal* (*timelike*) if it admits a representative belonging to J (rep. I). An Ad-orbit of $\mathfrak{iso}(1,3)$ is *causal* (*timelike*) if it admits an element belonging to $L(J)$ (resp. $L(I)$). An Ad-orbit is an *horismos* Ad-orbit if it is causal but not timelike.

The logarithm of an element belonging to $ISO(1,3)^{\uparrow}$ gives those matrices of $\mathfrak{iso}(1,3)$ whose exponential gives the original matrix. This set is non-empty because $ISO(1,3)^{\uparrow}$ is exponential. Clearly, the logarithm sends conjugacy classes into unions of Ad-orbits.

The previous theorem implies

Corollary 4.19. *The exponential of a causal (timelike) orbit gives a causal (resp. timelike) conjugacy class. The logarithm of a causal (resp. timelike) conjugacy class is a union of causal (resp. timelike) Ad-orbits.*

We reformulate the relativistic Chasles' theorem emphasizing the physical content of the classification. For this reason we focus on the infinitesimal transformations of M whose exponential moves at least one point $x \in M$ into its causal future $J^+(x)$.

In what follows $\ln I$ ($\ln J$) denotes the subset of $\mathfrak{iso}(1,3)$ made of matrices whose exponential is contained in I (resp. J). Clearly, $L(I) \subset \ln I$, and analogously $L(J) \subset \ln J$.

Theorem 4.20 (Relativistic Chasles' theorem, causal Lie cone version)**.**

- *Let $\mathcal{P} \in \mathfrak{IL}$, $\mathcal{P} \neq 0$, and suppose that there is a point $q \in M$ such that $P(s) = \exp(\mathcal{P}s)$ sends q to its timelike future for some $s > 0$. Then it is possible to choose a reference frame such that $\mathcal{P}$ takes one of the following*

matrix forms

$$(a)\ \begin{pmatrix} 0 & -a & 0 & 0 & -1 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tau = (aK^1 + \omega J^1 + H)\tau, \quad \textit{where } a > 0,\ \omega \neq 0,$$

$$(b)\ \begin{pmatrix} 0 & 0 & -a & 0 & -1 \\ 0 & 0 & -\omega & 0 & 0 \\ -a & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tau = (aK^2 - \omega J^3 + H)\tau, \quad \textit{where } a, \omega \geq 0.$$

where $\tau > 0$*. Stated in another way, the orbits of* $\mathfrak{iso}(1,3)$ *under the Ad action of* $ISO(1,3)^{\uparrow}$ *which admit an element in* $\ln I$ *admit a representative which is either of type (a) (if* $I_2 \neq 0$*) or of type (b) (if* $I_2 = 0$*). The constants* a, ω, τ*, are arbitrary as long as they satisfy*

$$(a^2 - \omega^2)\tau^2 = -2I_1, \tag{28}$$

$$I_2 \neq 0 \ \Rightarrow \qquad a\omega\tau^2 = I_2, \tag{29}$$

$$I_2 = 0 \ \Rightarrow \qquad \omega^2\tau^4 = I_3, \tag{30}$$

$$I_1 = I_2 = I_3 = 0\ (F = 0) \ \Rightarrow \qquad \tau^2 = -I_4. \tag{31}$$

Whenever case (b) applies, it is possible to choose the frame in such a way that $0 \leq \omega \leq a$ *(if* $I_1 \leq 0$*) or* $a = 0$*,* $\omega = 2I_1/\sqrt{I_3}$ *(if* $I_1 > 0$*). We have pure rotation if* $I_2 = 0$*,* $I_1 > 0$ *or* $I_1 = I_2 = I_3 = 0$ *(*$F = 0$*). If pure rotation does not apply, then* $\tau > 0$ *can be chosen arbitrarily, and once this is done,* $|\omega|$ *and* a *are uniquely determined.*

- *Let* $\mathcal{P} \in \mathfrak{IL}$*, and suppose that there is a point* $q \in M$ *such that* $P(s) = \exp(\mathcal{P}s)$ *sends* q *to some point in* $J^+(q)\backslash\{q\}$ *for some* $s > 0$*, and that* $\mathcal{P}$ *does not have the property of the previous point. Then it is possible to choose a reference frame such that* $\mathcal{P}$ *takes one of the following matrix forms*

$$(c)\ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \lambda = (\omega J^1 - \epsilon_2 P^1 + H)\lambda \quad \begin{array}{l} \textit{where } \omega \geq 0,\ \lambda > 0, \\ \textit{and } \epsilon_2 = \pm 1, \end{array}$$

and where λ *and* ω *are arbitrary as long as they satisfy* $\lambda\omega = \sqrt{2I_1}$*, or*

$$(d)\ \begin{pmatrix} 0 & 0 & -\eta & 0 & -1 \\ 0 & 0 & -\eta & 0 & -1 \\ -\eta & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \lambda = [(K^2 - J^3)\eta - P^1 + H]\lambda \quad \begin{array}{l} \textit{where } \eta \neq 0, \\ \textit{and } \lambda > 0, \end{array}$$

and where η *and* λ *are arbitrary as long as they satisfy the constraints.*

Stated in another way, the orbits of $\mathfrak{iso}(1,3)$ *under the Ad action of* $ISO(1,3)^{\uparrow}$ *which admit an element in* $\ln J$ *but none in* $\ln I$*, admit representative (c) (if* $I_1 > 0$*) or (d) (if* $I_1 = 0$*).*

Proof. According to theorem 4.17 we can suppose that q is sent to its (timelike) causal future for every $s > 0$.

Let us choose a reference frame with origin at q and let $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ be the corresponding matrix form of $\mathcal{P}$. Let $\begin{pmatrix} \Lambda(s) & -\bar{b}(s) \\ \bar{0}^{\intercal} & 1 \end{pmatrix}$ be the matrix of $\exp(\mathcal{P}s)$. Since $\bar{b}(s)$ is timelike for every $s > 0$ we have $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix} \in L(I)$. By theorem 4.16 the frame can actually be chosen in such a way that $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ takes one of the forms 1,2,4,7,9, (with $\epsilon_1 = 1$) of theorem 4.7. We have to show that in each of these cases, through a suitable frame change, we can bring the matrix to forms (a) or (b).

We are going to show that we can obtain (a) starting from 1, and (b) from 2, 4, 7 or 9. In other words we get (a) if $I_2 \neq 0$ and (b) if $I_2 = 0$.

Thus let us suppose that $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ is the representative given in point 1, theorem 4.7. Let us observe that $\varphi, \theta \neq 0$. Since $\mathrm{Im}F \supset \mathrm{Span}(e_0, e_1)$, through a translation of the frame we reach the matrix form

$$\begin{pmatrix} 0 & -\varphi & 0 & 0 & -\tau \\ -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & 0 & 0 & -1 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tau,$$

where $\tau > 0$ can be chosen arbitrarily and $a = \varphi/\tau$, $\omega = \theta/\tau$.

Let us come to the cases that will lead us to the form (b).

In case 9 set $\tau = \sqrt{-I_4}$, $a = \omega = 0$.

Suppose that we are in case 2. Through translation of the frame we reach the matrix form

$$\begin{pmatrix} 0 & -\varphi & 0 & 0 & -c \\ -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where we can choose $c > |b|$. The next identity holds

$$\begin{pmatrix} \cosh\gamma & 0 & 0 & -\sinh\gamma & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\sinh\gamma & 0 & 0 & \cosh\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\varphi & 0 & 0 & -c \\ -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh\gamma & 0 & 0 & \sinh\gamma & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sinh\gamma & 0 & 0 & \cosh\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\varphi\cosh\gamma & 0 & 0 & -(c\cosh\gamma - b\sinh\gamma) \\ -\varphi\cosh\gamma & 0 & 0 & -\varphi\sinh\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi\sinh\gamma & 0 & 0 & (c\sinh\gamma - b\cosh\gamma) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Let $\tau > 0$ be defined by $\tau := \sqrt{c^2 - b^2}$. The freedom in c shows that $\tau > 0$ can be chosen arbitrarily. Since $c, \tau > 0$ we can choose γ such that $\tanh \gamma = b/c$, so that $c \sinh \gamma - b \cosh \gamma = 0$ and $c \cosh \gamma - b \sinh \gamma = \tau$. Thus defining $a = \varphi c/\tau^2 > 0$ and $\omega = \varphi b/\tau^2$ we obtain (observe that $c = \tau/\sqrt{1 - (\omega/a)^2}$ and $\frac{b}{c} = \frac{\omega}{a}$)

$$\begin{pmatrix} 0 & -a & 0 & 0 & -1 \\ -a & 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tau,$$

which through a suitable rotation of the reference frame can be brought to the form

$$\begin{pmatrix} 0 & 0 & -a & 0 & -1 \\ 0 & 0 & -\omega & 0 & 0 \\ -a & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tau.$$

We observe that in this case $0 \le \omega < a$. For future reference we record that the original matrix can be rewritten

$$\begin{pmatrix} 0 & -\varphi & 0 & 0 & 0 \\ -\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a/\gamma & 0 & 0 & 0 \\ -a/\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\omega/a)\gamma \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tau, \qquad (32)$$

where $\gamma := 1/\sqrt{1 - (\omega/a)^2}$.

In case 7 we first boost the frame in the plane $\mathrm{Span}(e_0, e_1)$ and make a translation so that $\alpha = 1$ gets replaced by any chosen $\alpha > 0$ and the entry $-\sqrt{I_3}$ gets replaced by $-\sqrt{I_3}/\alpha$. We define $a = \omega = \alpha^2/\sqrt{I_3}$, and $\tau = \sqrt{I_3}/\alpha$. We observe that the common module of a and ω can be chosen freely due to the freedom in α.

Suppose that $\begin{pmatrix} F & -\bar{w} \\ \bar{0}^{\intercal} & 0 \end{pmatrix}$ is the representative given in point 4, theorem 4.7, with $\epsilon_1 = 1$. Let us observe that $\varphi = 0$; $\theta, b \neq 0$. Defined $\tau = b$, $a = 0$ and $\omega = \theta/b = 2I_1/\sqrt{I_3}$, after a rotation of the frame we obtain the matrix form (b) with $a = 0$, $\omega \neq 0$.

So far all the cases that we have considered that lead to case (b) with $a, \omega \neq 0$ show that we can always satisfy the inequality $0 \le \omega \le a$. Of all the cases that we have considered just case 4 gives $a < \omega$, but we can regard it as a case of aligned angular velocity and acceleration. In any case, it is convenient to observe that case 4 can be brought to the form (b) with a, ω such that $0 < a < \omega$, through a sequence of translation, boost and rotation following calculations similar to those of case 2.

The statement concerning Eqs. (28)-(31) can be easily checked calculating the invariants for (a) and (b).

The last point is an easy consequence of theorem 4.16, through inspection of cases 5,11 with $\epsilon_1 = 1$, and 8 with $I_4 = 0$, of theorem 4.7.

□

4.7 Lorentzian extension of Chasles' theorem

We are ready to prove that any orientation and time orientation preserving isometry of Minkowski spacetime which sends some point to its chronological future, can be accomplished through the dragging of spacetime points by the motion of an observer's reference frame, where the observer moves with constant acceleration and angular velocity for some proper time interval.

Theorem 4.21. *(Relativistic Chasles' theorem, group version, timelike part) Suppose that $P : M \to M$, $P \in IL_+^\uparrow$, sends some point to its chronological future, then there is a reference frame on M with respect to whose coordinates P takes one of the following matrix forms*

$$(a) \qquad \exp[(aK^1+\omega J^1+H)\tau] = \exp(\begin{pmatrix} 0 & -a & 0 & 0 & -1 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\tau)$$

$$= \begin{pmatrix} \cosh(a\tau) & -\sinh(a\tau) & 0 & 0 & -\frac{1}{a}\sinh(a\tau) \\ -\sinh(a\tau) & \cosh(a\tau) & 0 & 0 & \frac{1}{a}[\cosh(a\tau)-1] \\ 0 & 0 & \cos(\omega\tau) & \sin(\omega\tau) & 0 \\ 0 & 0 & -\sin(\omega\tau) & \cos(\omega\tau) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(b) \qquad \exp[(aK^2-\omega J^3+H)\tau] = \exp(\begin{pmatrix} 0 & 0 & -a & 0 & -1 \\ 0 & 0 & -\omega & 0 & 0 \\ -a & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\tau)$$

$$= \begin{pmatrix} 1+(a\tau)^2/2 & -(a\tau)^2/2 & -a\tau & 0 & -\tau-a^2\tau^3/6 \\ (a\tau)^2/2 & 1-(a\tau)^2/2 & -a\tau & 0 & -a^2\tau^3/6 \\ -a\tau & a\tau & 1 & 0 & a\tau^2/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(c) \qquad \exp[(aK^2-\omega J^3+H)\tau] = \exp(\begin{pmatrix} 0 & 0 & -a & 0 & -1 \\ 0 & 0 & -\omega & 0 & 0 \\ -a & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\tau)$$

$$= \left(\begin{matrix} 1+\gamma^2[\cosh(a\tau/\gamma)-1] & -(\omega/a)\gamma^2[\cosh(a\tau/\gamma)-1] \\ (\omega/a)\gamma^2[\cosh(a\tau/\gamma)-1] & 1-(\omega/a)^2\gamma^2[\cosh(a\tau/\gamma)-1] \\ -\gamma\sinh(a\tau/\gamma) & (\omega/a)\gamma\sinh(a\tau/\gamma) \\ 0 & 0 \\ 0 & 0 \end{matrix}\right. \cdots$$

$$\cdots \left.\begin{matrix} -\gamma\sinh(a\tau/\gamma) & 0 & (\omega/a)^2\gamma^2\tau-\frac{1}{a}\gamma^3\sinh(a\tau/\gamma) \\ -(\omega/a)\gamma\sinh(a\tau/\gamma) & 0 & (\omega/a)\gamma^2\tau-(\omega/a^2)\gamma^3\sinh(a\tau/\gamma) \\ \cosh(a\tau/\gamma) & 0 & \frac{1}{a}\gamma^2[\cosh(a\tau/\gamma)-1] \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right),$$

where $\gamma(a,\omega) := 1/\sqrt{1-(\omega/a)^2}$, $\tau > 0$ and, furthermore, in (a) $a \ge 0$, in (b) $a = \omega \neq 0$, in (c) $0 \le \omega < a$. The arbitrariness in a, ω, τ, is the same as that given in theorem 4.20.

Proof. By theorem 4.2 $ISO(1,3)^{\uparrow}$ is exponential, thus there is some $\mathcal{P} \in \mathfrak{IL}$ such that $P = \exp \mathcal{P}$. The remainder of the theorem follows from theorem 4.20 after some algebra (the last matrix can also be obtained through a transformation of the frame from Eq. (32)). □

The previous theorem involves the exponential of elements of $\mathfrak{iso}(1,3)$. The reader interested in general closed exponentiation formulas is referred to [47, 41, 15].

Theorem 4.22. *(Relativistic Chasles' theorem, group version, horismos part) Suppose that $P : M \to M$, $P \in IL_+^{\uparrow}$, sends some point $q \in M$ to some point in $J^+(q)\backslash\{q\}$, but none to its chronological future, then there is a reference frame on M with respect to whose coordinates P takes one of the following matrix forms*

$$(a) \qquad \exp[(\omega J^1 - \epsilon_2 P^1 + H)\lambda] = \exp(\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\lambda)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & -\lambda \\ 0 & 1 & 0 & 0 & -\epsilon_2\lambda \\ 0 & 0 & \cos(\omega\lambda) & \sin(\omega\lambda) & 0 \\ 0 & 0 & -\sin(\omega\lambda) & \cos(\omega\lambda) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(b) \qquad \exp\{[(K^2 - J^3)\eta - P^1 + H]\lambda\} = \exp(\begin{pmatrix} 0 & 0 & -\eta & 0 & -1 \\ 0 & 0 & -\eta & 0 & -1 \\ -\eta & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\lambda)$$

$$= \begin{pmatrix} 1 + (\eta\lambda)^2/2 & -(\eta\lambda)^2/2 & -\eta\lambda & 0 & -\lambda \\ (\eta\lambda)^2/2 & 1 - (\eta\lambda)^2/2 & -\eta\lambda & 0 & -\lambda \\ -\eta\lambda & \eta\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\lambda \geq 0$ and, moreover, in (a) $\omega > 0$, $\epsilon_2 = \pm 1$, $\lambda\omega = \sqrt{2I_1}$. The arbitrariness in λ, ω, η, is the same as that given in theorem 4.20.

Proof. By theorem 4.2 $ISO(1,3)^{\uparrow}$ is exponential, thus there is some $\mathcal{P} \in \mathfrak{IL}$ such that $P = \exp \mathcal{P}$. The remainder of the theorem follows from theorem 4.20 after some algebra. □

The transformations of type (a) might be called *lightlike screws*. We mention that in [43, 39] the term *screw* is used for what we call *roto-boost*. Since *roto-boosts* appear already in the study of the Lorentz group, which does not include translations, it seems to be inappropriate to use the term *screw* for those transformations.

Remark 4.23. With reference to the canonical motions (a), (b) and (c) of theorem 4.21, it is interesting to calculate the position $\Lambda^{-1}\bar{b}$ of the frame with

respect to its coordinates at time $\tau = 0$. They are

$$(a) \quad \begin{pmatrix} \frac{1}{a}\sinh(a\tau) \\ \frac{1}{a}[\cosh(a\tau)-1] \\ 0 \\ 0 \end{pmatrix}, \qquad (b) \quad \begin{pmatrix} \tau + a^2\tau^3/6 \\ a^2\tau^3/6 \\ a\tau^2/2 \\ 0 \end{pmatrix},$$

where we omit the expression for (c) which is complex and not particularly illuminating. It seems curious that we get a rather simple polynomial expression for case (b) which corresponds to equal and orthogonal acceleration and angular velocity.

We end this work giving in table 7 and 8 the classification of timelike and horismos Ad-orbits of $\mathfrak{iso}(1,3)$. There we choose the simplest representative which, however, might not belong to $L(I)$ (resp. $L(J)\backslash L(I)$). Nevertheless, we keep the parametrization as it is inherited by its conjugacy equivalent which belongs to $L(I)$ (resp. $L(J)\backslash L(I)$). The last column reminds us that once the parameters selecting the orbit have been fixed, the freedom left in the choice of simplifying reference frame selects some characteristic geometric object. These ingredients provide the generalization to the relativistic case of Mozzi and Chasles' instantaneous axis of rotation.

5 Conclusions

We have generalized Chasles' theorem to the Lorentzian spacetime case, proving that every inhomogeneous proper orthochronous Lorentz transformation, which sends some point to its chronological future, can be obtained through the displacement of an observer which moves at constant angular velocity and constant acceleration (theorems 4.20 and 4.21). We have also given an horismos version of this result in which a lightlike geodesic plays the role of the observer's worldline (theorem 4.22).

Intuitively, this result states that if the isometry satisfies the mentioned causality requirement, then it is generated through some canonical frame motion along the natural causal entities that live on spacetime: observers and light rays.

In order to accomplish this result we first proved the exponentiality of the proper orthochronous inhomogeneous Lorentz group (Theor. 4.2). We studied the Lie algebra introducing a complete set of Ad-invariants (Theor. 4.4) which allowed us to classify the Ad-orbits (Theor. 4.7). As a corollary, we obtained a classification of the adjoint inequivalent Killing fields of Minkowski spacetime (Theor. 4.7, Cor. 4.8).

It is clear that space translations, while being isometries, are not generated by any observer's causally meaningful motion. In order to obtain a relativistic version of Chasles' theorem it was necessary to impose some causality condition. The weakest is the requirement that the transformation sends some point to its chronological (causal) future. Keeping this observation in mind we went to study the causal semigroup of the inhomogeneous Lorentz group and its Lie cone. In this respect, we connected this weak causality condition with the

apparently stronger condition which wants the logarithm of the transformation on the Lie wedge 4.17, and we identified those Ad-orbits that admit a causal representative (Theor. 4.16). Finally, we proved the relativistic generalization of Chasles' theorem.

In our analysis we payed special attention to the geometrical content of the Lorentz transformations, summarizing the possibilities in tables 7 and 8. Given the conjugacy class (or Ad-orbit) and the appropriate geometric information, it is then possible to fully recover the transformation and, more importantly, to grasp its physical meaning.

Table 7: Relativistic Chasles' theorem and reconstruction (timelike Lie wedge version). The simplest representatives here displayed are not necessarily those belonging to $L(I)$, nevertheless they are parametrized keeping in mind the physical interpretation of their equivalents which belong to $L(I)$.

Type	Families of timelike orbits (Def. 4.18) (some matrices are given up to a positive factor)	Parameters (omitted positive factor)	Description	Geometric ingredients
(p1)	$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	[*none*]	timelike translation (inertial motion)	timelike direction
(p2)	$\begin{pmatrix} 0 & -a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$a > 0$, $\omega \neq 0$.	acceleration aligned with angular velocity	oriented timelike 2-plane with origin
(p3)	$\begin{pmatrix} 0 & -a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$a > 0$.	acceleration	oriented spacelike 2-plane
(p4)	$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\omega > 0$,	rotation	oriented timelike 2-plane and timelike direction on it
(p5)	$\begin{pmatrix} 0 & -a/\gamma & 0 & 0 & 0 \\ -a/\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\omega/a)\gamma \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\gamma := 1/\sqrt{1-(\omega/a)^2}$ $a > 0$, $0 < \omega < a$.	the acceleration and angular velocity are orthogonal	oriented spacelike 2-plane and oriented spacelike direction on it
(p6)	$\begin{pmatrix} 0 & 0 & -a & 0 & -1 \\ 0 & 0 & -\omega & 0 & 0 \\ -a & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$a = \omega > 0$.	the acceleration and angular velocity are orthogonal	oriented lightlike 2-plane

Table 8: Relativistic Chasles' theorem and reconstruction (horismos Lie wedge version)

Type	Families of horismos orbits (Def. 4.18)	Parameters	Description	Geometric ingredients
(p7)	$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \lambda$	$\epsilon_2 = \pm 1$, $\lambda > 0$	(positive/negative helicity) lightlike screw	oriented timelike 2-plane
(p8)	$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	[none]	[none]	oriented lightlike 2-plane and f.d. lightlike vector on it

References

[1] Abraham, G.: Classes of the n-dimensional Lorentz group. Proc. Ind. Acad. Sci. Sect. A **28**, 87–93 (1948)

[2] Alexandrov, A. D.: On Lorentz transformations. Uspekhi Mat. Nauk. **5(3)**, 187–193 (1950). (in Russian)

[3] Alexandrov, A. D.: Mapping of spaces with families of cones and space-time transformations. Annali di Matematica Pura e Applicata **103**, 229–257 (1975)

[4] Ball, R. S.: *The Theory of Screws: A study in the dynamics of a rigid body.* Dublin: Hodges, Foster & Co. (1876)

[5] Barbot, T.: Globally hyperbolic flat space-times. J. Geom. Phys. **53**, 123–165 (2005)

[6] Carmeli, M.: *Group theory and general relativity.* New York: McGraw-Hill Book Company (1977)

[7] Ceccarelli, M.: Screw axis defined by Giulio Mozzi in 1763 and early studies on helicoidal motion. Mechanism and Machine Theory **35**, 761–770 (2000)

[8] Coll, B. and Martínez, F. S.: Composition of Lorentz transformations in terms of their generators. Gen. Relat. Gravit. **34**, 1345–1356 (2002)

[9] Coll, B. and San José, F.: On the exponential of the 2-forms in relativity. Gen. Relat. Gravit. **22**, 811–826 (1990)

[10] Dimentberg, F. M.: *The screw calculus and its applications in mechanics.* U.S. Department of Commerce, NTIS, AD-680 993 (1968)

[11] Doković, D. Ž. and Hofmann, K. H.: The surjectivity question for the exponential function of real Lie groups: a status report. Journal of Lie theory **7**, 171–199 (1997)

[12] Duistermaat, J. J. and Kolk, J. A. C.: *Lie groups.* Berlin: Springer (2000)

[13] Duval, C., Burdet, G., Künzle, H. P., and Perrin, M.: Bargmann structures and Newton-Cartan theory. Phys. Rev. D **31**, 1841–1853 (1985)

[14] Duval, C., Gibbons, G., and Horváthy, P.: Celestial mechanics, conformal structures, and gravitational waves. Phys. Rev. D **43**, 3907–3922 (1991)

[15] Fredsted, J.: Exponentiation of the spinor representation of the Lorentz group. J. Math. Phys. **42**, 4497–4502 (2001)

[16] Gallier, J.: Notes on differential geometry and Lie algebras (2012). Lecture notes

[17] Hall, G. S.: *Simmetries and curvature structure in general relativity*. Singapore: World Scientific (2004)

[18] Hawking, S. W. and Ellis, G. F. R.: *The Large Scale Structure of Space-Time*. Cambridge: Cambridge University Press (1973)

[19] Hilgert, J., Hofmann, K. H., and Lawson, J. D.: *Lie groups, convex cones, and semigroups*. Oxford: Claredon Press (1989)

[20] Hilgert, J. and Need, K.-H.: *Lie semigroups and their applications*. Berlin: Springer-Verlag (1993)

[21] Hofmann, K. H. and Ruppert, W. A. F.: Lie groups and subsemigroups with surjective exponential function. Memoirs of the American Mathematical Society **130**, 1–174 (1997)

[22] Kobayashi, S. and Nomizu, K.: *Foundations of Differential Geometry*, vol. I of *Interscience tracts in pure and applied mathematics*. New York: Interscience Publishers (1963)

[23] Lomony, J.: Classes of the homogeneous Lorentz group. Bulletin of Cal. Math. Soc. **55**, 51–55 (1963)

[24] Minguzzi, E.: Classical aspects of lightlike dimensional reduction. Class. Quantum Grav. **23**, 7085–7110 (2006)

[25] Minguzzi, E.: A geometrical introduction to screw theory. Eur. J. Phys. (2013). To appear in the January issue. arXiv:1201.4497

[26] Misner, C. W., Thorne, K. S., and Wheeler, J. A.: *Gravitation*. San Francisco: Freeman (1973)

[27] Moskowitz, M.: Correction and addenda to: On the surjectivity of the exponential map for certain Lie groups. Annali di Matematica pura ed applicata **173**, 351–358 (1999)

[28] Murray, R. M., Li, Z., and Sastri, S. S.: *A mathematical introduction to robotic manipulation*. Boca Raton, Florida: CRC Press (1994)

[29] Naber, G. L.: *The geometry of Minkowski spacetime*. New York: Springer-Verlag (1992)

[30] Needham, T.: *Visual complex analysis*. Oxford: Claredon Press (1997)

[31] Nishikawa, M.: On the exponential map of the group $O(p,q)_0$. Memoirs of the Faculty of Science, Kyushu University, Ser. A **37**, 63–69 (1983)

[32] Penrose, R.: The apparent shape of a relativistically moving sphere. Proc. Camb. Phil. Soc. **55**, 137–139 (1959)

[33] Penrose, R. and Rindler, W.: *Spinors and Space-Time vol 1*. Cambridge: Cambridge University Press (1984)

[34] Riesz, M.: *Clifford Numbers and Spinors*. New York: Springer (1993)

[35] Schwartz, H. M.: An extension of Euler's theorem to Minkowski space. Am. J. Phys. **31**, 864–867 (1963)

[36] Schwartz, H. M.: A further note on an extension of Euler's theorem to Minkowski space. Am. J. Phys. **33**, 376–378 (1965)

[37] Selig, J. M.: *Geometric fundamentals of robotics*. New York: Springer (2005)

[38] Sexl, R. U. and Urbantke, H. K.: *Relativity, groups, particles: special relativity and relativistic symmetry in field and particle physics*. Wien: Springer-Verlag (2001)

[39] Shaw, R.: The conjugacy classes of the homegeneous Lorentz group. Quarterly Journal of Mathematics **20**, 333–345 (1969)

[40] Shaw, R. and Bowtell, G.: The bivector logarithm of a Lorentz transformation. Quarterly Journal of Mathematics **20**, 497–503 (1969)

[41] Silva Leite, F. and Crouch, P.: Closed forms for the exponential mapping on matrix Lie groups based on Putzer's method. J. Math. Phys. **40**, 3561–3568 (1999)

[42] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., and Herlt, E.: *Exact solutions of Einsteins field equations*. Cambridge: Cambridge University Press (2003)

[43] Synge, J. L.: *Relativity: the special theory*. Amsterdam: North-Holland Publishing Co. (1956)

[44] Weinberg, S.: *The Quantum Theory of Fields*, vol. I. Cambridge: Cambridge University Press (1995)

[45] Wigner, E.: On unitary representations of the inhomogeneous Lorentz group. Ann. Math. **40**, 149–204 (1940)

[46] Zeeman, E. C.: Causality implies the Lorentz group. J. Math. Phys. **5**, 490–493 (1964)

[47] Zeni, J. R. and Rodrigues Jr., W. A.: The exponential of the generators of the Lorentz group and the solution of the Lorentz force equation. Hadronic Journal **13**, 317–323 (1990)